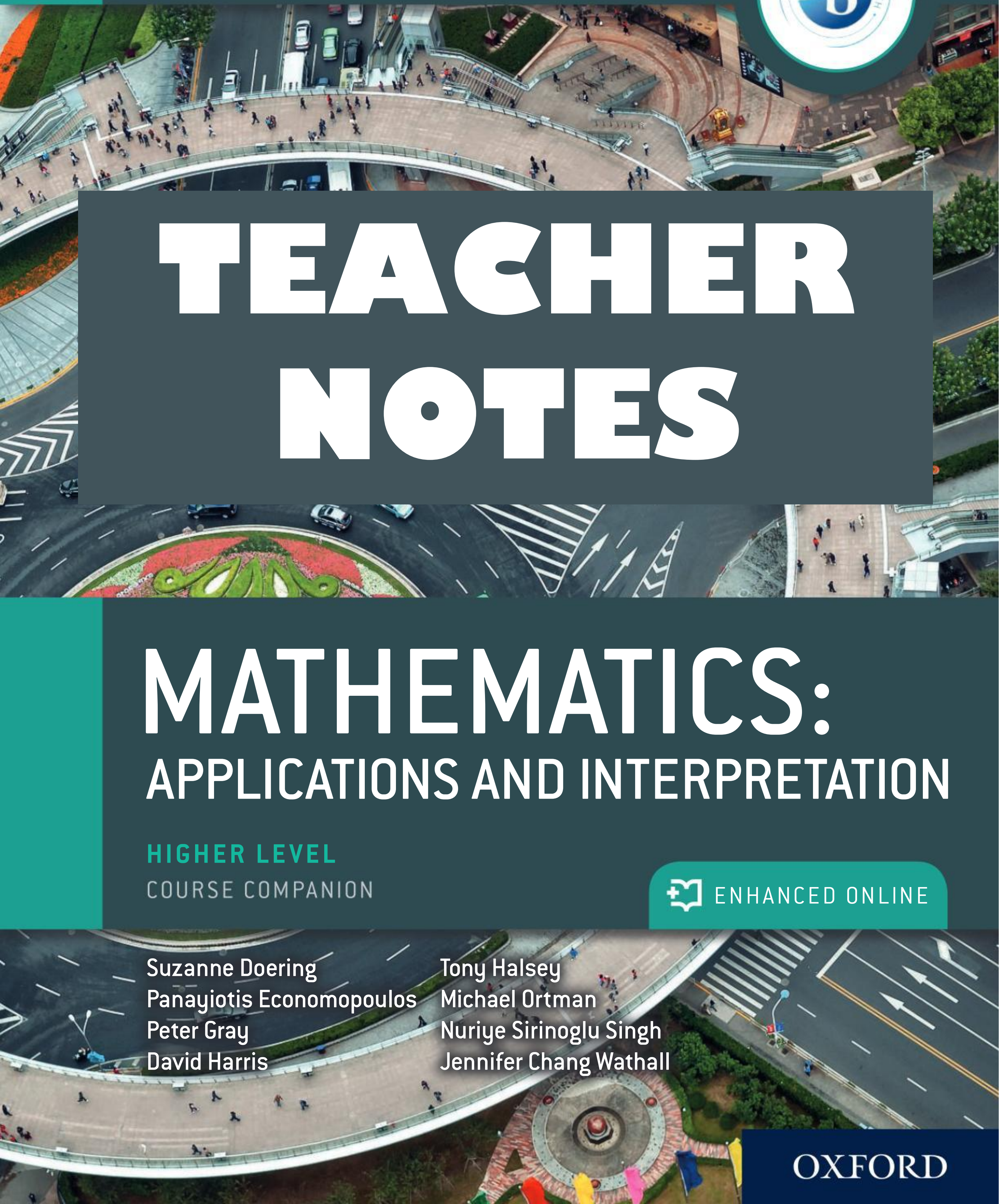
OXFORD IB DIPLOMA PROGRAMME



TEACHER NOTES

MATHEMATICS:
APPLICATIONS AND INTERPRETATION

HIGHER LEVEL

COURSE COMPANION

ENHANCED ONLINE

Suzanne Doering
Panayiotis Economopoulos
Peter Gray
David Harris
Tony Halsey
Michael Ortman
Nuriye Sirinoglu Singh
Jennifer Chang Wathall

OXFORD

1 Measuring space: accuracy and geometry

Essential understandings

Geometry and trigonometry allow us to quantify the physical world, enhancing our spatial awareness in two and three dimensions. This branch provides us with the tools for analysis, measurement and transformation of quantities, movements and relationships.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- The properties of shapes are highly dependent on the dimension they occupy in space.
- Volume and surface area of shapes are determined by formulae, or general mathematical relationships or rules expressed using symbols or variables.
- The relationships between the length of the sides and the size of the angles in a triangle can be used to solve many problems involving position, distance, angles and area.
- Different representations of trigonometric expressions help to simplify calculations.
- Systems of equations often, but not always, lead to intersection points.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	Investigation
An estimate approximates a value of a measurement, to allow simpler calculations or to check whether calculated results make sense.	Investigation 1
Percentage error allows us to find the magnitude of the error relative to the size of the measured length and to compare measurement errors. Measurement errors can be compared by expressing them first as a percentage of the measured property.	Investigation 2
Standard form more easily indicates the relative size of very large and very small numbers and leads to more efficient calculations.	Investigation 3
In any triangle, the ratio of the sine of any angle to the side opposite the angle remains a constant.	Investigation 4
The ambiguous case of the sine rule considers two possible triangles once given two sides and one non-included angle, with the unknown angle positioned opposite the longer of the two sides.	Investigation 5
For any triangle, the area can be found by forming right-angled triangles and using the sine ratio to find the perpendicular height.	Investigation 6
Surface area can be calculated as a sum of lateral areas and bases. Prisms and cylinders contain two bases and rectangular lateral area and pyramids and cones, contain one base and a triangular lateral area.	Investigation 7

Syllabus sections covered in this chapter:

- SL1.5*
- SL1.1*
- SL1.6
- SL3.1*
- SL3.2*
- SL3.3*
- SL3.4
- AHL1.10

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 2: Measuring space: accuracy and geometry	Page 11: Example 5 Page 26: Example 14 Page 34: Example 19	Page 14: Example 7 Page 18: Example 9 Page 20: Example 11 Page 24: Example 13	Pages 12, 26, 35

Assessment opportunities

End of chapter summary	Chapter review	Exam-style questions
Page 36	Page 39	Page 40

1.1 Representing numbers exactly and approximately

Investigation 1

Conceptual understanding:

An estimate approximates a value of a measurement, to allow simpler calculations or to check whether calculated results make sense.

1 165 cm, assuming that the books are roughly the height of Hamilton, and she is an average woman's height.

2 16 500 pages. Estimation could be often done by sampling. For example, we could stack pages of paper and see, for example, how many pages are in a stack that is about 10 cm high. Or measure the height of a standard pack of paper for printing containing 500 – it's about 5 cm high, which means that a stack of 1000 pages is about 10 cm high. A 165 cm-tall stack then will contain about 16.5 × 1000, or 16 500 pages.

3 **Factual:** What is an estimate? What is estimation? How would you go about estimating? How can comparing measures help you estimate?

Answer: An estimate approximates a value of a measurement – as close as possible – to allow simpler calculations or to check whether calculated results make sense. Estimation is often done by comparing the attribute that we measure to another one, or by sampling.

4 **Conceptual:** Why are estimations useful?

Answer: An estimate approximates a value of a measurement, to allow simpler calculations or to check whether calculated results make sense.

TOK

Words in English can have different meanings. How else can we interpret "error"? Does error mean that there was a mistake?

What other examples of misleading mathematical words do you know?

Investigation 2;

Conceptual understandings:

Percentage error allows us to find the magnitude of the error relative to the size of the measured length and to compare measurement errors.

Measurement errors can be compared by expressing them first as a percentage of the measured property.

1 92.44cm − 91.44 cm = 1 cm, 31.48 cm – 30.48 cm = 1 cm

2 Both measurements have the same absolute error. However, the difference of 1 cm in a true measurement of 30.48 cm is relatively bigger than a difference of 1 cm compared to 91.44 cm.

3 $\frac{1}{91.44}$ or 1.09%

4 $\frac{1}{30.48}$ or 3.28%

5 **Conceptual:** In what ways is expressing measurement errors as a percentage of the measured length helpful?

Answer: Percentage error allows us to find the magnitude of the error relative to the size of the measured length and to compare measurement errors.

6 **Conceptual:** How can measurement errors be compared?

Answer: Measurement errors can be compared by expressing them first as a percentage of the measure property.

TOK

In many cases, our measurements and calculations include errors. Very often courts rely on interpretations of forensic data and invite experts to courtrooms to give their opinion. If the percentage error of a certain DNA testing or drug testing is 0.0526, can we be certain the person in question is guilty? What would be considered as an acceptable error rate, especially when the stakes are so high?

IM

You might want to have students research the history of number from the Sumerians and its development to the present Arabic system; this is a fascinating development to trace. You could go back to the Ishango bone, evidence of counting from 20 000 years ago.

Investigation 3

Conceptual understanding:

Standard form more easily indicates the relative size of very large and very small numbers and leads to more efficient calculations.

1 **a** India: 1.34×10^9, USA: 3.24×10^8, China: 1.41×10^9

b Look at powers of ten first.

2 **a** 6×10^{14} **b** 8×10^5 **c** 8×10^{14} **d** 1.5×10^{19} **e** 3.6×10^{-1}

3 If $bc < 10$ then answer is $bc \times 10^{m+n}$. If $bc \geq 10$ then answer is $(bc \div 10) \times 10^{m+n+1}$.

4 The powers of 10 are combined using laws of exponents.

5 **a** and **b**

Country	GDP per capita (estimate)	Population (estimate)	GDP (estimate)
India	2×10^3	1×10^9	2×10^{12}
USA	6×10^4	3×10^8	$18 \times 10^{12} = 1.8 \times 10^{13}$
China	9×10^3	1×10^9	9×10^{12}

c India 2.66×10^{12} USA 1.93×10^{13} China 1.22×10^{13}

d India and US had the same magnitude, China did not. China's population was rounded more than the other two (1.41 to 1 billion).

6 **Conceptual:** How does standard form help with calculations?

Answer: Standard form more easily indicates the relative size of very large and very small numbers and leads to more efficient calculations.

TOK

Some questions for debate:

- Can we hide facts by rounding?
- Do we always need an exact answer?
- What are the advantages/disadvantages of rounding?

Rounding to the nearest thousand or million dollars in large financial transactions is done to make it easier for those involved to simplify the figures, but what happens to the "rounded off" money?

1.2 Angles and triangles

Investigation 4

Conceptual understanding:

In any triangle, the ratio of the sine of any angle to the side opposite the angle remains a constant.

Part **1** All 3 ratios are equal.

Part **2** All 3 ratios are equal.

Part **3** All 3 ratios are equal.

Part **4** **Conceptual:** What can you say about the ratio of the sine of an angle to the length of the side opposite the angle, in any triangle?

Answer: In any triangle, the ratio of the sine of any angle to the side opposite the angle remains a constant.

It most useful to use the sine rule with the angles 'on top' to find a missing angle when you know one angle and two sides. It most useful to use the sine rule with the side lengths 'on top' to find a missing side when you now one side and two angles.

TOK

An Einstein quote: "Mathematical systems are invented, but it is a matter of discovery which of the various systems apply to reality. You can invent any formal system and prove theorems from axioms with complete certainty."

An axiom is a statement that is considered to be true and does not require a proof and is the starting point of reasoning. Axioms are then used to prove other statements using logical deduction. Euclidean geometry is an example of an axiomatic system.

Investigation 5

Conceptual understanding:

The ambiguous case of the sine rule considers two possible triangles once given two sides and one non-included angle, with the unknown angle positioned opposite the longer of the two sides.

1 54.0°

2 Angle D is obtuse in the diagram, but the measurement obtained is acute.

3 Students should verify.

4 **a** The base angles of an isosceles triangle are congruent.

b They are supplementary angles (sum to 180°).

5 126.0°

6 141°, 180° − x

7 The sine rule gives 20.4° so obtuse answer would be 159.6°. This is not possible because angles of a triangle add up to 180°.

8 **Conceptual:** Why does the sine rule not always have just one solution?

Answer: The ambiguous case of the sine rule considers two possible triangles once given two sides and one non-included angle, with the unknown angle positioned opposite the longer of the two sides.

Investigation 6

Conceptual understanding:

For any triangle, the area can be found by forming right angled triangles and using the sine ratio to find the perpendicular height.

1 A = $\frac{c \times h}{2}$

2 $\frac{h}{a} = \sin \angle B$, $h = a \times \sin \angle B$

3 $A = \frac{c \times a \times \sin \angle B}{2}$

4 **Factual:** What is the formula for the area of any triangle?

Answer: $A = \frac{c \times a \times \sin \angle B}{2}$

5 **Conceptual:** How can you find the area of any triangle?

Answer: For any triangle, the area can be found by forming right-angled triangles and using the sine ratio to find the perpendicular height.

TOK

There are sometimes misconceptions that arise through the wording. This is a chance to see the place of language in the communication of mathematics.

It feels right to instinctively say that the angles in a triangle sum to 180°. However, when a triangle is drawn on a basketball, the lines are not straight and the angles no longer sum to 180°.

After this analysis, you might conclude that reason and instinct can act alone, but it is more when instinct and reason work together to create the best possible result.

Developing inquiry skills

Look again at the opening problem. The Great Trigonometric Survey used an instrument called a theodolite to measure the angles from points A and B to the summit E.

Angle BAE = 30.5° and angle ABE = 26.2°.

Points A and B are 33 km apart. Find the height of Mount Everest to the nearest metre.

Answer: First find AE. To do this, find the angle opposite the known side: $\hat{E} = 180 - 30.5 - 26.2 = 123.3°$

Then substitute into the sine rule: $\frac{AE}{\sin 26.2°} = \frac{33}{\sin 123.3°}$ so AE = 17.4 km

Now find the height *h* from point E to [AB]: sin30.5° =angle ABE

so *H* = 8.8473 km =8847 m

1.3 Three-dimensional geometry

TOK

Start by showing this quote from Galileo:

"Philosophy is written in this grand book, the Universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed.

"It is written in the language of mathematics, and its characters are triangles, circles and other geometric figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth."

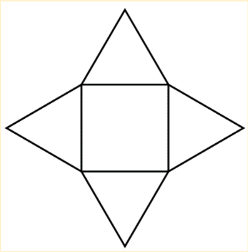
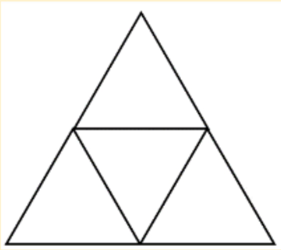
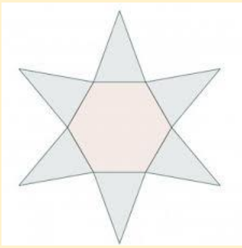
Ask students for their response to the quote.

You might want to consider exploring how the Platonic solids govern the structure of any given atom.

Investigation 7

Conceptual understanding:

Surface area can be calculated as a sum of lateral areas and bases. Prisms and cylinders contain two bases and rectangular lateral area and pyramids and cones, contain one base and a triangular lateral area.

1 **a**  **b**  **c** 

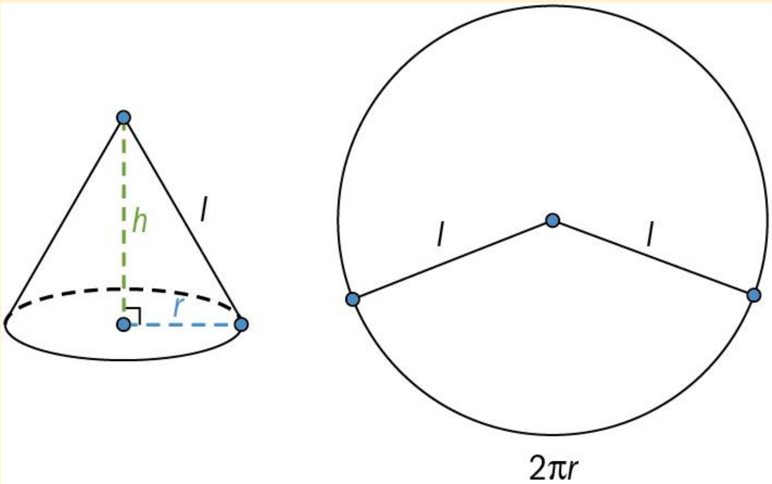
2 Let *n* be the number of triangles (= number of sides of the base), *b* the base of each triangle (length of one side of the base), and *l* the height of each triangle (slant height of the pyramid).

Then *LA* = sum of areas of triangle = number of triangles $\times \frac{1}{2}bh = \frac{1}{2}(nb)l = \frac{1}{2}pl$ where *p* is the perimeter of the base.

3 Students should identify how their variables relate to the variables in the formula.

4 No – the slant heights differ if the base is not regular.

5 a



b i radius is l

ii circumference $= 2\pi l$ arc of circle = circumference of cone = $2\pi r$

c $\frac{LA}{\pi l^2} = \frac{2\pi r}{2\pi l}$ so LA = πrl

6 a Factual: What are the formulae for finding surface area for these solids?

Answer: Surface area = lateral area + base

b Conceptual: How are the formulae for surface area derived?

Answer: Surface area can be calculated as a sum of lateral areas and bases.

c Conceptual: What is the same about finding the surface area of various solids? What is different?

Answer: Surface area can be calculated as a sum of lateral areas and bases. Prisms and cylinders contain two bases and rectangular lateral area and pyramids and cones, contain one base and a triangular lateral area.

Developing inquiry skills

Look back at the opening problem. The radius of Mount Everest is approximately 16 km, and the average snow depth is approximately 4 m. Estimate the amount of snow at Mount Everest.

Possible solution: Modeling Everest as a cone with radius 16 000 m, Everest would have a volume of

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(16000)^2(8847)$$
$$= 2.3717265 \times 10^{12}\text{m}^2$$

The snow being 4 m deep adds approximately 4 m to the base and to the height of the cone, and the snow is the difference between this cone and the mountain:

$$V_{\text{snow}} = \frac{1}{3}\pi(16000)^2(8851) - = 2.3717265 \times 10^{12} = 2.26 \times 10^9\text{m}^3$$

So, there is approximately 2.3 billion cubic metres of snow.

Squares

Approaches to Learning/learner profile: Research, Critical Thinking

Exploration Criteria: Personal engagement (C), Use of mathematics (E)

IB Topic: Proof, Geometry, Trigonometry

Proof is an important topic in Mathematics. The emphasis in the Mathematics: Applications and Interpretation course is on applying rather than proving so this could act as an opportunity to see proof in action that the students will not otherwise get.

Proof is a difficult topic to pursue for an exploration as it will not always score highly in all criteria. Some suggestions are given at the end of how to possibly extend this and other topics which give the possibility of scoring better on Personal engagement (Criterion C) and even Reflection (Criterion D).

The problem

Some history:

This problem first appeared in the "Mathematical Games" column of Scientific American in 1970. It was submitted by Martin Gardner who had been introduced to the problem by a friend, Lyber Katz, who had to do it for extra credit in Moscow when he was in grade 4. The problem was published again in the *Journal of Recreational Math* in 1971, where readers could try to solve it and send back their solutions. Many people, ranging from high school students to undergraduates and people in mathematics professions, sent back many different solutions. Charles W. Trigg gathered all of these solutions and published 54 different solutions under the title "A three-square geometry problem" in the next volume of the *Journal of Recreational Math*.

Emphasize to students that these 54 solutions, and others found since, lead to exactly the same answer. In mathematics class, students tend to focus on just finding the correct solutions rather than concentrating on how to find that answer. To find a solution one way may be easy, but to find another alternative solution to the same problem perhaps requires a deeper understanding of it, different skills and mathematical concepts. The idea of finding multiple solutions also links to the concept of divergent thinking, which is a crucial thought process used by mathematicians, engineers, architects and other problem solvers.

This is an interesting problem – not because the proofs of the answer are particularly difficult, but because there are so many of them!

Exploring the problem

This is an interesting crossover with TOK that looks at inductive and deductive reasoning and the concept of proof in mathematics and all other areas of knowledge.

The answer is 90°, and students will likely get something close to this if they measure the angles or just guess.

Direct proof

$\alpha = 45°$

$\alpha = \beta + \phi = 45°$

$AC = \sqrt{2}$, $AD = \sqrt{5}$, $AE = \sqrt{10}$

To help students explain why the triangles are similar, ask:

- Which sides correspond?

SSS similar triangle theorem

In ΔACD, the ratio of CD : AC : AD = 1 : √2 : √5

In ΔACE, the ratio of AC : CE : AE = √2 : 2 : √10 = 1(√2) : √2(√2) : √5(√2)

So CD : AC : AD = AC : CE : AE

Hence ∠CAD = ∠CEA using properties of the two similar triangles.

Using exterior angle theorem:

∠ACB = ∠CAD + ∠ADC

and so $\alpha = \beta + \phi$

and $\beta + \phi = 45°$

and so $\alpha + \beta + \phi = 90°$

Proof using an auxiliary line

∠BAC = α: Angle of isosceles right triangle.

∠EAF = ϕ: Alternate interior angles.

This will complete the proof because it will show that ∠BAC + ∠EAF + ∠GAC = 90° since ∠BAF is a right angle because it is an angle of a square.

To help students show that ΔGAC and ΔABD are similar, you could hint:

Calculate the lengths of sides CG and AC.

CG : AC = AB : BD since $\frac{\sqrt{2}}{2} : \sqrt{2} = 1 : 2$

and ∠GCA = ∠ABD = 90°

∠GAC = ∠BDA = β

These are equivalent angles in similar triangles.

Completing the proof:

∠BAC + ∠EAF + ∠GAC = 90°

and so $\alpha + \beta + \phi = 90°$

Adding an auxiliary line is often a good method in a geometrical proof.

Proof using the cosine rule

∠XEY = β because it is a 2 by 1 triangle like ΔABD.

AE = √10 and AY = √5

52 = (√10)2 + (√5)2 − 2(√10)(√5)cos θ

$\cos\theta = \frac{-1}{\sqrt{2}}$ so $\theta = 135°$

$\beta + \phi = 180 - 135 = 45°$

To complete the proof:

$\alpha = 45°$

So $\alpha + \beta + \phi = 90°$

You could ask:

Which proof do you think is the best? Which do you like the most?

What criteria are you using to make these judgements?

Again, this is a good TOK point where the concept of 'best' can be discussed.

Students might consider best to be, for example:

- quickest or easiest
- most efficient or most elegant
- most surprising or using unusual mathematics
- hardest to understand, etc.

Questions involving proof are a good starting point for an exploration, but would rarely score highly if that is all it involves, as there is limited chance for personal engagement beyond possibly engaging in mathematics that is new to the student or beyond the scope of the course.

Extension

You could direct students towards problems in the textbook or problems on sites such as Nrich, Brilliant, or various mathematics competition sites.

Students could combine this with the extension task in Chapter **14** on Spearman's Rank and provide their classmates with different proofs and ask them to rank them.

The problems that students choose may also give them a chance to extend the problems and ask them to consider 'what if…?' to demonstrate Personal engagement (Criterion C).

For the problem in this task, they could discuss:

- How else could the problem be extended?
- What about 4 squares, 5 squares, *n* squares?

Students could then work on generalizing the problem.

This is good advice whenever a student tackles a problem where solutions are readily available by an internet search.

2 Representing and describing data: descriptive statistics

Essential understandings

Statistics is concerned with the collection, analysis and interpretation of data and the theory of probability can be used to estimate parameters, discover empirical laws, test hypotheses and predict the occurrence of events. Statistical representations and measures allow us to represent data in many different forms to aid interpretation.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Organizing, representing, analysing and interpreting data, and utilizing different statistical tools facilitates prediction and drawing of conclusions.
- Different statistical techniques require justification and the identification of their limitations and validity.
- Approximation in data can approach the truth but may not always achieve it.
- Correlation and regression are powerful tools for identifying patterns and equivalence of systems.
- Modelling and finding structure in seemingly random events facilitates prediction.
- Statistical literacy involves identifying reliability and validity of samples and whole populations in a closed system.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
In general, discrete data occurs when the data has been counted and continuous data occurs when the data has been measured.	Investigation 1
We need to take a sample from a large population as there would be too much work involved in performing analysis of such a large set of data. It should be noted that in other cases, taking the whole population is not just unwieldy, but often impossible. If one group is underrepresented in the population, it will probably be underrepresented in the sample, and could possibly be missed altogether. Students should develop an understanding that bias in the collection of data may affect the results and reducing bias allows a sample to represent the general population. Each method of sampling will have its own advantages and disadvantages, but if they are to provide meaningful insight into the general population, then inherent biases must be considered.	Investigation 2
Real-world statistics uses technology for number crunching, and statistical understanding for interpretation because there are very extensive calculations to be done which would be very time-consuming to do by hand, but technology can do the number crunching very easily.	Investigation 3

Adding a constant to every value in a data set increases the mean by that constant, and has no effect on the standard deviation as the actual spread between data points stays the same.	Investigation 4
Data may be represented to give a misleading impression by altering scales or only selecting certain data points.	Investigation 5
Cumulative frequency graphs allow for estimation of median, quartiles and percentiles, rather than just providing an interval in which they lie.	Investigation 6
Extrapolation predicts values outside the range of data available, and assumes that the pattern shown in the data continues outside that range and this may be risky.	Investigation 7

Syllabus sections covered in this chapter:

- SL1.2*
- SL4.1*
- SL4.2*
- SL4.3*
- SL4.4*

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 44: Representing and describing data: descriptive statistics	Page 51: Example 2 Page 60: Example 6 Page 67: Example 8	Page 51: Example 2 Page 53: Example 3 Page 59: Example 5 Page 68: Example 9	Pages 50, 58, 64, 71

Assessment opportunities

End of chapter summary	Chapter review	Exam-style questions
Page 72	Page 75	Page 77

Developing inquiry skills

Write down any similar inquiry questions you might ask if you were asked to investigate the relationship between two different quantities, for example, GDP per capita and infant mortality or life expectancy and population.

What questions might you need to ask in these scenarios which differ from the scenario where you are investigating the relationship between GDP and life expectancy?

Answer: Where would I be able to find the data necessary for analysis? How might I best display the data, either for analysis, or for a visual impression? What might I want to do with any findings? How reliable are my findings likely to be?

2.1 Collecting and organizing data

International-mindedness

Who else could be considered "the father of statistics"?

Investigation 1

Conceptual understanding:
In general, discrete data occurs when the data has been counted and continuous data occurs when the data has been measured.

1 **Factual:** Is the data discrete or continuous?

Answer: The data is discrete as the IB only awards integer grades. There is no grade 4.7 in HL Maths.

2

Grade, g	3	4	5	6	7
Frequency	1	5	2	6	4

3 **Factual:** Is the data discrete or continuous?

Answer: Continuous

4

Height, h, in cm	$2.5 \le h < 3.5$	$3.5 \le h < 4.5$	$4.5 \le h < 5.5$	$5.5 \le h < 6.5$	$6.5 \le h < 7.5$
Frequency	1	5	2	6	4

5 **Conceptual:** When would discrete or continuous data occur? Think about how you would obtain the data and whether you would need any particular tools.

Answer: In general, discrete data occurs when the data has been counted and continuous data occurs when the data has been measured.

6 **Factual:** Is the data discrete or continuous?

Answer: Continuous.

7 The interval for each age would be from the day of his/her birthday until the day of his/her next birthday.

Height, h, in cm	$3 \le h < 4$	$4 \le h < 5$	$5 \le h < 6$	$6 \le h < 7$	$7 \le h < 8$
Frequency	1	5	2	6	4

Investigation 2

Conceptual understandings:

We need to take a sample from a large population as there would be too much work involved in performing analysis of such a large set of data. It should be noted that in other cases, taking the whole population is not just unwieldy, but often impossible.

If one group is underrepresented in the population, it will probably be underrepresented in the sample, and could possibly be missed altogether.

Students should develop an understanding that bias in the collection of data may affect the results and reducing bias allows a sample to represent the general population.

Each method of sampling will have its own advantages and disadvantages, but if they are to provide meaningful insight into the general population, then inherent biases must be considered.

1 The head teacher would not want to publish any data with names of students. Also, if there were names attached there might be temptations to add additional interpretations.

2 We need to take a sample from the population as there would be too much work involved in performing analysis of such a large set of data.

3 **Conceptual:** Why do we need to take a sample from the population?

Answer: We need to take a sample from a large population as there would be too much work involved in performing analysis of such a large set of data. It should be noted that in other cases, taking the whole population is not just unwieldy, but often impossible.

4 In making samples, different students, or groups of students, will obtain different samples. For each of methods 2, 3 and 4, there will need to be further decisions as to how to obtain the random sample. Whether to stratify the boys and girls separately? Whether to sample 15 boys then 15 girls, or keep selecting at random until you reach a quota? Students should be encouraged to consider these difficulties.
Students should develop an understanding that sampling involves analyzing data from a sample in order to draw conclusions about the whole population. This will be elaborated through the coming sections.
5 **Factual:** Does your sample have an equal number of Robotics Club and Astronomy Club members?

Answer: Differing answers for different samples, but this might be just one bias highlighted in the collection of data.

6 **Factual:** Does your sample have representative numbers from the different years?

Answer: As above. Differing answers for different samples, but this might be just one bias highlighted in the collection of data.

7 **Conceptual:** Can you identify any bias created when you have used this method of sampling?

Answer: If one group is underrepresented in the population, it will probably be underrepresented in the sample, and could possibly be missed altogether.

Students should develop an understanding that bias in the collection of data may affect the results and reducing bias allows a sample to represent the general population.

8 **Conceptual:** Can you think of different occasions where this type of sampling could have created extreme bias?

Answer: For example, if students were arranged in order of house, with the house captain first and there were 18 students in each house, then you could choose all house captains or none.

9 For each student the results would be different.

Conceptual: Can you identify any additional bias that might have been created by the sampling method or a bias that has been eliminated?

Answer: For method 2 there is a reduction of bias in terms of other factors that might be happening in different year groups (for example changes in curriculum or staffing), whereas in looking at just one year group, factors that might be peculiar to that year group will be emphasised. For method 3, you will be sure not to have more of one club than the other, but that could be disproportionate with the numbers in the school. It should be considered that stratifying in an attempt to reduce bias can in fact have quite the opposite effect.

10 $\frac{69}{545}\times 30 = 3.8$, $\frac{104}{545}\times 30 = 5.7$, $\frac{95}{545}\times 30 = 5.2$, $\frac{136}{545}\times 30 = 7.48$, $\frac{141}{545}\times 30 = 7.8$ so numbers round to 4, 6, 5, 7, 8.

11 $\frac{545}{30} = 18.2$ so taking every 18th student will mean an even spread for the sample. Taking every tenth student would lead to disproportionate numbers from particular years. Using this method in years where there are more students, there will automatically be more students in the sample as every 18th student will fall more often.

12 Many discussions possible, much of which has been discussed above. There are good arguments for justifying the representativeness of each

13 **Conceptual:** What are the different sampling methods and how do they provide meaningful insight into the general population?

Answer: Each method of sampling will have its own advantages and disadvantages, but if they are to provide meaningful insight into the general population, then inherent biases must be considered.

Developing inquiry skills

In the opening section you saw data from different countries regarding life expectancy, GDP, population and region. If you were to take a sample from all those countries to draw conclusions, how might you stratify your sample in order to reduce bias?

Answer: One way might be to divide the graph up into regions, perhaps with diagonal lines, and take one country at random from each region. There are many other possibilities.

2.2 Statistical measures

TOK

In question **4** and the associated discussion box, you might want to look at number sets where the mean, mode and median are different, and ask which is the best measure to use.

Consider the responses in terms of the perspectives of the different people.

You might want to consider the readings on reliability in statistics.

TOK

The sample and population are at work here. You might want to look into when we should use Greek letters like μ and when we should use the symbols from the modern English alphabet like $\bar{x}$.

Investigation 3

Conceptual understanding:
Real-world statistics uses technology for number crunching, and statistical understanding for interpretation because there are very extensive calculations to be done which would be very time-consuming to do by hand, but technology can do the number crunching very easily.

1 Results will be different for each student.

2 **Conceptual:** Explain why you would have used technology for all those calculations.

Answer: Real-world statistics uses technology for number crunching, and statistical understanding for interpretation because there are very extensive calculations to be done which would be very time-consuming to do by hand, but technology can do the number crunching very easily.

Investigation 4

Conceptual understanding:
Adding a constant to every value in a data set increases the mean by that constant, and has no effect on the standard deviation as the actual spread between data points stays the same.

	Mean	**Standard deviation**
A	9	4.64
B	12	4.64
C	7	4.64
D	14	4.64
E	27	14.87
F	−18	9.27
G	4.5	2.32

1 **Factual:** What happens to the mean when you add or subtract a number from each term?

Answer: You add or subtract that number from the mean.

2 **Factual:** What happens to the standard deviation when you add or subtract a number from each term?

Answer: It remains unchanged.

3 **Factual:** What happens to the mean when you multiply each number by a constant?

Answer: You multiply that number by mean.

4 **Factual:** What happens to the standard deviation when you multiply each number by a constant?

Answer: You multiply the absolute value of that number by the standard deviation.

5 **Conceptual:** Why does adding a constant to every value in a data set result in no change in the standard deviation?

Answer: Adding a constant to every value in a data set increases the mean by that constant, and has no effect on the standard deviation as the actual spread between data points stays the same.

6 **a** 13 and 1.5 **b** 40 and 6

TOK

A typical teacher-led discussion might go something like this:

Over the centuries people have debated whether mathematics is discovered, or if it is simply invented by the minds of great mathematicians.

- What do you think?
- If you think it is discovered, where are you looking?
- If you think it is invented, why can't a mathematician say that he has 4 times 2 =10?
- Now, what about the standard deviation?

Developing inquiry skills

Take a sample from the data in the opening section and use it to calculate statistical summaries, to determine whether different regions have different life expectancies.

Answer: It will be different for each sample, but you would expect the results to show differences. For example, life expectancy in Europe is higher than most and life expectancy in Africa is lower from looking at the graphic.

2.3 Ways in which you can present data

TOK

Misleading statistics; examples of problems caused by absence of representative samples, e.g. Google flu trends, US presidential elections in 1936, *Literary Digest* v George Gallup, Boston "pot-hole".

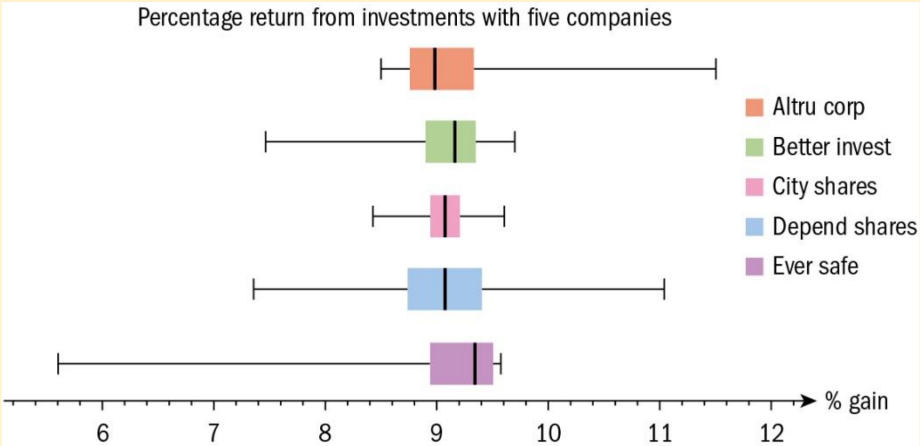
Ben Goldacre writes: "In 1954 a man called Darrell Huff published a book called *How to Lie with Statistics*. Chapter one is called 'The sample with built-in bias'.

Huff sets up his headline: 'The average Yaleman, Class of 1924, makes $25,111 a year!' said *Time Magazine*. That figure sounded pretty high: Huff chases it, and points out the flaws. How did they find all these people they asked? Who did they miss? Losers tend to drop off the alma mater radar, whereas successful people are in *Who's Who* and the College Record. Did this introduce selection bias into the sample? And how did they pose the question? Can that really be salary rather than investment income? Can you trust people when they self-declare their income? Is the figure spuriously precise?" And so on.

Investigation 5

Conceptual understanding:
Data may be represented to give a misleading impression by altering scales or only selecting certain data points.

1 The Box plot for the five companies shows

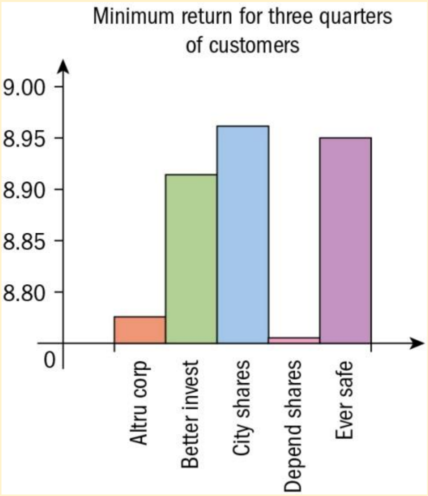


Various valid comments are possible. Altrucorp has chances of high returns with a low minimum return. However,it also has the lowest median so half the customers get a lower rate than elsewhere. Betterinvest is negatively skewed so has nearly 75% of customers with over about 9% return. Cityshares are symmetrical and with low deviation so a safe reliable investment. Dependshares are also symmetrical but greater deviation, meaning possibilities of higher returns, but also lower returns. Eversafe has nearly 75% of customers over 9% return, but a risk of very low return The median return for Eversafe is over 9.35, so the Eversafe claim is correct.

2 Seeing the box plot for Altrucorp with outliers marked highlights that it is very unlikely to get high returns. This might put you off taking the risk with the company.

3 The bar chart is very misleading as it distorts the difference in the mean return. It looks as though Altrucorp gives returns of four times that of Cityshares, which is not at all the case. This is because the axis does not start at zero. However, it is not dishonest as the axes are clearly labelled.

4 Cityshare might say something like "over 75% of customers having returns of over 8.95%". they might also use the number of investors in the company.



Conceptual: In what ways could you use graphics to mislead?

Answer: Data may be represented to give a misleading impression by altering scales or only selecting certain data points.

Investigation 6

Conceptual understanding:

Cumulative frequency graphs allow for estimation of median, quartiles and percentiles, rather than just providing an interval in which they lie.

1 19 caterpillars are between 3.5 and 4.5 cm and a further 56 caterpillars are between 4.5 and 5.5 cm. So, in total, there are 19 + 56 = 75 caterpillars less than 5.5 cm long.

2

Length, l (cm)	$l < 3.5$	$l < 4.5$	$l < 5.5$	$l < 6.5$	$l < 7.5$	$l < 8.5$	$l < 9.5$
Cumulative frequency	0	19	75	149	194	199	200

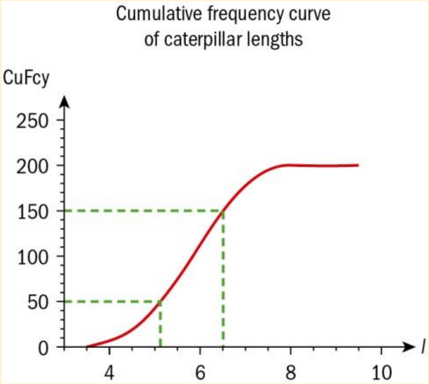
Factual: How are all the values for the cumulative frequency calculated?

Answer: Either by adding up all the frequencies up to that point, or by adding the frequency to the previous cumulative sum.

3 Only discrete data is entered into the calculator and it can only find answers for the median and quartiles accordingly. As all data in the interval [5.5,6.5[was entered as 6, no matter where the data fell in that interval, either early or late in the count, the calculator will read the answer as 6.

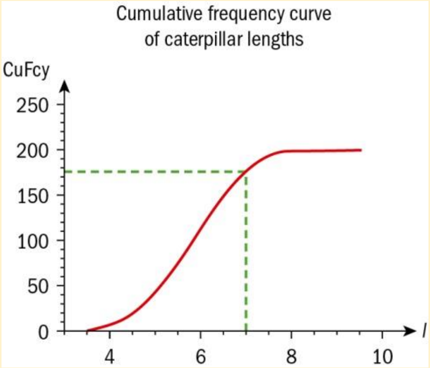
4 **Factual:** How would you use the cumulative frequency graph to find an estimate for the upper quartile of the data? Similarly use the graph to find the value of Q3 and of the interquartile range.

Answer: $Q_1 = 5.1$. Read across from 150, which is three quarters of the way through the data.



From the curve, $Q_3 = 6.5$ and $Q_1 = 5.1$ giving $IQR = 1.4$.

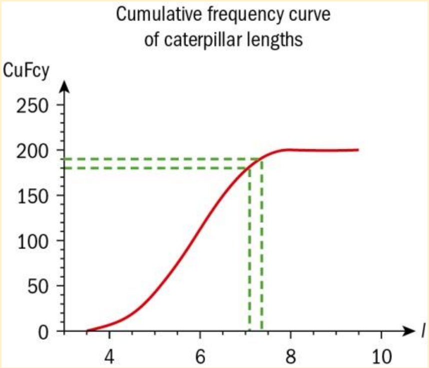
5 We can read up from 7 cm to find that about 176 caterpillars are less than 7 cm and so $200 - 176 = 24$ caterpillars are more than 7 cm.



6 **Conceptual:** What is the purpose of using cumulative frequency graphs?

Answer: Cumulative frequency graphs allow for estimation of median, quartiles and percentiles, rather than just providing an interval in which they lie.

7 $P_{95} = 7.35$ and $P_{90} = 7.1$ (found from reading from 190 and 180 respectively).



TOK

Teachers might want to consider the history and relevant "newness" of statistics. The amount of numeracy, tables, formulas and charts put statistics into most school mathematics curricula, but as statistics has become more important, its connections with everyday human sciences, natural science, arts and languages suggest teaching statistics across the curriculum might be more appropriate.

This is a good place for teachers to view the knowledge framework for mathematics.

Developing inquiry skills

Would you have a better understanding of a set of data after looking at either:

a the raw data

b summary statistics

c statistical charts

d a combination of two or more of the above

At the end of section 2.2, you collected a sample from the data in the opening section. Use that data to draw statistical charts to illustrate your findings.

Answer: Histograms, Box-plots, and cumulative frequency curves can all be used, and will be different for each student as they will have used different samples.

International-mindedness

A stimulating TED Talk from Hans Rosling at https://youtu.be/usdJgEwMinM

Claire Provost, who interviewed Rosling in 2013, said: "Given the timing, with all the talk about fake news, Hans Rosling stood for the exact opposite – the idea we can have debates about what could or should be done, but that facts and an open mind are needed before informed discussions can begin."

2.4 Bivariate data

TOK

The median, and the upper quartile divide the ordered data into four groups with approximately the same number of observations in each group.

- How can you do this for a small sample size like 2.5, 3.1, 6?

For small samples, there is no obvious way to do this, and concessions of some sort must be made.

- Do the quartiles have any meaning for samples of this size?

There are different algorithms in technology simply because different people have different ideas how to make the compromises. They may have slightly different objectives in mind how to use the quartiles in practice.

Respond to the question "To what extent can we rely on technology to produce our results?"

TOK

A common misconception is to consider them to mean the same thing.

Data is raw, unorganized facts or numbers.

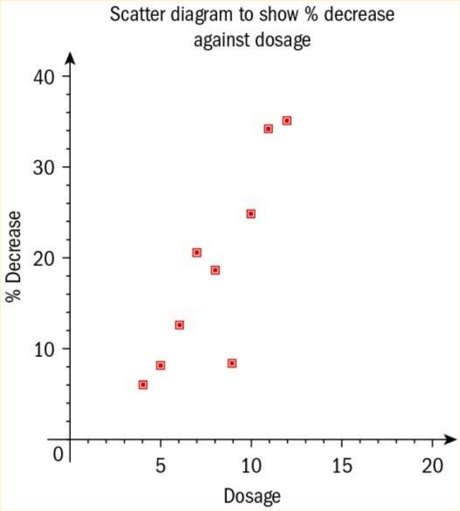
When you organise data or present it in a given context so as to make it useful, it is called information.

For example, the individual students' scores on a test are data but the class mean is information.

Investigation 7

Conceptual understanding:
Extrapolation predicts values outside the range of data available, and assumes that the pattern shown in the data continues outside that range and this may be risky.

1



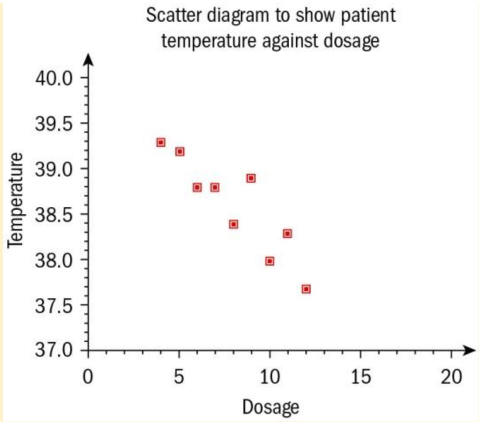
2 Yes, higher values of the dosage seem to correspond with higher values of the percentage decrease.

3 The dosage is what we control, and we hope that the percentage decrease will be as a result of the application of the drug. However, we cannot be sure that the drug is causing the reduction.

4 It appears that the point (9, 8.4) is an outlier as it does not seem to fit with the trend of the other points. Various reasons as to why this might occur, e.g. the patient forgetting to take the drug, some resistance the patient has to the treatment, the patient's drug being defective etc.

5 With this data point excluded, $r = 0.979$ indicating very high correlation.

6

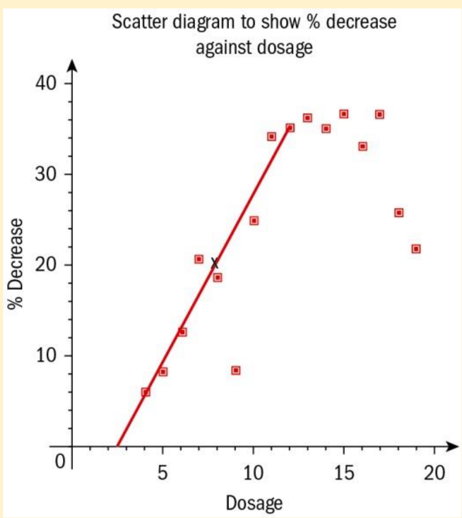


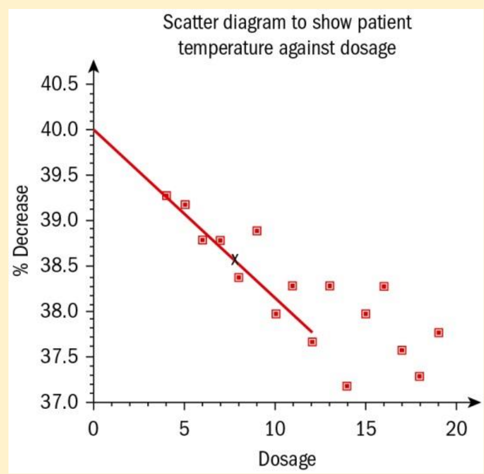
7 Excluding the outlier (not so obvious in this case), $r = -0.957$ indicating very strong negative correlation.

8 It is clear that the drug may not be causing the reduced temperature. For example, it may well be that the reduced infection itself causes the reduction in temperature, rather than directly the drug.

9 The two mean points are (7.875, 20.025) and (7.875, 38.5625)

10





11 It is clear that the trend does not continue in either case. Maybe above a certain point the body resists, or the disease develops immunity.

12 As it would be more difficult to fit a straight line through the data, we would expect a lower value of r.

13 **Conceptual:** Can we use extrapolation to predict beyond the data points? Why or why not?

Answer: Extrapolation predicts values outside the range of data available, and assumes that the pattern shown in the data continues outside that range: this may be risky.

Developing inquiry skills

For the country data given in the opening of this chapter, take a suitable sample to determine whether GDP and income have any correlation.

Answer: All samples will be different, but you would expect to see some correlation between the quantities.

What's the difference?

Approaches to Learning: Thinking Skills, Communicating, Collaborating, Research

Exploration Criteria: Presentation (A), Mathematical Communication (B) Personal Engagement (C), reflection (D), Use of Mathematics (E)

IB Topic: Statistics, Mean, Median, Mode, Range, Standard deviation, Box plots, Histograms

Many Internal Assessments use statistics. The basis of such explorations is often when the student makes and tests a hypothesis by collecting data. The data is partially analysed using some the techniques in this chapter – further useful techniques are introduced in chapter **14**. Many of these explorations take the form of a scientific enquiry and so this is a good opportunity for the teacher to discuss the use of Mathematics as a tool in other subjects. In this task these skills are developed further.

The student is required to make decisions about the data that they need to collect to meet the aim of the question they pose themselves. They can then analyse the data that is collected, and decide whether the analysis supports the hypothesis.

Students could prepare a mini-exploration (couple of pages only), poster or presentation to demonstrate what they have found out, students could then discuss what methods of data collection, analysis and representation are appropriate and effective. This will obviously require a longer amount of time allocated to it in class, but given that this is a common exploration topic then this may be worthwhile. It may be necessary, and indeed desirable, depending on different class situations for students to work in pairs, groups or as a whole class. It is certainly a good opportunity for collaboration between classmates as they discuss appropriate methods, analysis, conclusions and limitations.

The task could be assessed against a shortened form of the real IA assessment criteria (rather than all of it as that can be a little daunting – a suggestion is given below). So this task offers a good opportunity to introduce or revisit the criteria

Criterion A: Presentation (2)

Your writing should be well-organised, coherent, logically developed and easy to follow. It includes an aim (hypothesis) and conclusion.

Criterion B: Mathematical Communication(2)

You should use appropriate mathematical language, define key terms and use appropriate forms of mathematical representation.

Criterion C: Personal Engagement (2)

Rationale, ideas expressed mathematical ideas in your own way, experiment conducted.

Criterion D: Reflection (2)

You should review, analyse and evaluate. You should consider the significance, consider possible limitations and/or extensions, consider the accuracy of results, consider the appropriateness of your approach.

Criterion E: Use of Mathematics (2)

The mathematics that you use should be correct.

Demonstrate a reaction time test (or similar) to the class. Ask students to discuss in pairs groups or as a class what could possibly make your performance better or worse if you were to do it again. Ask why that may be the case. Ask how you could test it.

Here are some sites that can collect data:

- Reaction timer – nrich
- Multitasking – notdoppler
- Reaction timer – faculty.washington.edu

- Reaction timer – faculty.washington.edu
- Reaction timer – faculty.washington.edu
- Reaction timer – faculty.washington.edu
- Reaction timer – faculty.washington.edu
- Eye-balling game – woodgears
- The sheep dash game – BBC
- Subitizing – BBC
- Estimating angles – nrich
- Estimating time – nrich

It is possible to conduct experiments without technology as well:

- Standing on one foot without and then with eyes closed or before and after exercise.
- Testing reaction times by dropping and catching a ruler between two fingers.

Raghu runs an experiment with a group of 25 students

Each member of a group of 25 students does a reaction test and Raghu records their times to complete it. Raghu wishes to test the change to the reaction time if the experiment was run again.

Discuss the following:

- Do you think that the performance in the group would improve, stay the same or get worse if they used their non-dominant hand instead?
- they did the test at a different time of day, say early in the morning or late at night?
- they were now allowed to practice for a while and were then asked to do it again?
- they were being observed doing the test the second time?

Raghu could instead compare the group results with another group. Do you think that a different group would have different results if

- they were younger or older?
- the two groups were comprised a different gender?
- they were given different instructions?
- they were given some form of motivation?

The discussions should hopefully make the next part of the task more straightforward and help students to come up with ideas of their own. They can, of course, also use the example provided.

Your task is to devise an experiment to test a hypothesis you are going to make. In the example above reaction times have been used. There are many different sites that can be used to test reaction times. However there are also good online sites that test different skills and abilities such as estimating angle sizes or times. There are also others that test ability to multitask or spacial awareness. If the experiment required to collect the data you want does not exist then you could devise it yourself.

Step 1: Decide on what you are going to test. State aim and hypothesis

Write down the aim of the experiment you are going to do and hypothesis on the result.

- Why do you think this is important? What are the implications of the results that you may find in a broad sense?
- Are you testing for a change in the average? Or for the spread of the data? Or a change in the distribution of the data? Or some combination? Make sure it is clear.

For example, Raghu could write the following (Note these are brief responses only at this stage – in a real exploration it would be possible and necessary to expand on these with more detail):

- The aim of my exploration is to investigate the effect of the time of day on the reaction time of grade 12 students at my school.
- My hypothesis is that if students do the same reaction test in the morning and then again in the afternoon that the general performance would be worse in the afternoon because the students are likely to be more tired.

This is important as it has implications to when students are given or asked to perform particular tasks that require reactions and concentration.

Step 2: Consider how the data is going to be collected and write a plan

You may need to consider some of the following:

- What resources/site will be required?
- Will the data be sufficient? How many people/students will you be able to and need to collect data from to mean that your results and conclusions will be statistically valid?
- Exactly what data do you need to collect?
- How are you going to record the data you collect?
- Have you prepared a results sheet? Have you run a trial experiment?
- Are there any biases in the way the experiment is presented?
- How do you ensure that everyone gets the same instructions?
- Is your experiment a justifiable way of analysing your hypothesis? What is the justification?
- What are the possible criticisms? Can you do anything about these?
- Is the experiment reliable? Would someone else likely reach a similar conclusion to you if they used the same method?

Students should share their aim, hypothesis, plans and responses to these questions with someone else in their class or with a group or the whole class before they begin collecting any data. They can then discuss what is good and what might need improving.

Step 3: Run the experiment and collect the data

Be prepared with a results sheet to collect the data.

Give clear, consistent instructions and ensure that everyone knows what they are doing before starting.

It may be advisable that only a couple of tests are chosen from the class to conduct; otherwise there are too many tests going on at once and it can get very complicated. Students could vote somehow on which of the problems they wish to pursue and work collectively as a class or small groups to overcome any of the difficulties of data collection that might arise. If this is the case, it may be necessary, if possible, to arrange to use a different class or classes to do the experiment depending on the nature of the task, as by discussing the question itself they may be biasing the results in their own class. This is also interesting for discussion.

It might not be possible for this to be done in a single lesson so homework/pre-planning may be required.

Step 4: Present the data for comparison and analysis

- How are you going to present the data so that the two sets can be easily compared? Use some of the ideas in this chapter (tables, box plots, histograms)
- How are you going to summarise the location and/or the spread of the two sets of data so that you can compare them?
- Do you need to find all of the summary statistics covered in this chapter? Which ones are appropriate for your aim and hypothesis?

Students will need to be aware of the difference between the mean and standard deviation of a population and that of a sample. This is not covered in this chapter but will be an important distinction to make in an exploration.

Step 5: Comparison and analyse

- Describe the differences between your two sets of data.
- Make sure to keep this relevant to the aim and hypothesis stated at the beginning. Only calculate what is needed. For example, if not interested in spread of data then there is no need to calculate the standard deviation. However, this might be useful when discussing comparisons of means, in which case you will need to find it.

Step 6: Conclusions and implications

- So what are the conclusions from the experiment? Are they different or the same as your hypothesis? To what extent? Why?
- How confident are you in your results? How could you be more certain?
- What is the scope of your conclusions? How far reaching could you take your conclusions or are they limited to a particular school, age group, gender, location etc?
- How have your ideas changed since your original hypothesis?

Discussing the implications and scope of a student's results is essential to this task. They will need to understand the nature of inductive reasoning here and that there conclusions are only valid for their particular experiment although possible to speculate beyond the scope of their investigation this is not going to be conclusive.

Some possible extensions:

It is possible to test data using a more 'mathematical' test. You may like to investigate the 'difference in means test' for example.

This is covered later in the course and in chapter **14** of this book.

- How could you test whether the spread of the class data has changed significantly rather than just the average performance?

There are statistical tests that can be found for this.

- In what ways could you incorporate the work you have done so far on bivariate data?

If the students have collected two sets of data for each student then it will be possible to plot a scatter diagram of the data and see if a correlation exists.

3 Dividing up space: coordinate geometry, Voronoi diagrams, vectors, lines

Essential understandings

Geometry and trigonometry allow us to quantify the physical world, enhancing our spatial awareness in two and three dimensions. This branch provides us with the tools for analysis, measurement and transformation of quantities, movements and relationships.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- The properties of shapes are highly dependent on the dimension they occupy in space.
- Volume and surface area of shapes are determined by formulae, or general mathematical relationships or rules expressed using symbols or variables.
- Different representations of trigonometric expressions help to simplify calculations.
- Systems of equations often, but not always, lead to intersection points.
- In two dimensions, the Voronoi diagram allows us to navigate, path-find or establish an optimum position.
- Vectors allow us to determine position, change of position (movement) and force in two and three-dimensional space.
- Matrices are a form of notation which allow us to show the parameters or quantities of several linear equations simultaneously.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
The properties of shapes depend on the dimension they occupy in space. The relationships between the length of the sides and the size of the angles in a triangle can be used to solve many problems involving position, distance, angles and area.	Investigation 1
Equations of straight lines can be rearranged into equivalent equations, such as gradient–intercept form, general form, point–gradient form, all representing the gradient of the line and its position on the coordinate grid.	Investigation 2
All points equidistant from two fixed points lie on the perpendicular bisector of the line segment joining those two points.	Investigation 3
The edges of a cell on a Voronoi diagram lie along the perpendicular bisectors of the line segments joining the points.	Investigation 4
The incremental algorithm allows the edges of a Voronoi diagram to be easily found.	Investigation 5
The solution to the toxic waste dump problem lies at a vertex of the Voronoi diagram.	Investigation 6

Adding vectors gives the resultant effect of all the vectors acting successively.	Investigation 7
Vectors can be given as a magnitude and direction or written in component form, and using component form allows you to find the resultant easily.	Investigation 8
The scalar product can be used to find the angle between two vectors.	Investigation 9
The vector product of two vectors represents a vector perpendicular to the two vectors.	Investigation 10
The equation of a line $\mathbf{r} = \mathbf{a} + \mathbf{b}t$ is defined by the vectors **a** and **b** where **a** is the position vector of a point on the line and **b** is a vector in the direction of the line.	Investigation 11
The motion of a particle travelling with constant velocity can be given as the equation of a straight line with direction vector **v** and parameter t which represents the time from the initial position.	Investigation 12

Syllabus sections covered in this chapter:

- SL2.1*
- SL2.3*
- SL2.4*
- SL3.1*
- SL3.5
- SL3.6
- AHL3.10
- AHL3.11
- AHL3.12
- AHL3.13

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 82: Dividing up space: coordinate geometry, Voronoi diagrams, vectors, lines	Page 100: Example 4 Page 106: Example 5 Page 124: Example 14	Page 87: Example 1 Page 90: Example 2 Page 100: Example 4 Page 124: Example 14	Pages 85, 95, 101, 112, 118, 126

Assessment opportunities

End of chapter summary	Chapter review	Exam-style questions
Page 127	Page 130	Page 131

3.1 Coordinate geometry in 2 and 3 dimensions

TOK

Use the fact that the cosine rule is one possible generalization of Pythagoras' theorem to explore the concept of "generality".

Investigation 1

Conceptual understanding:

The properties of shapes depend on the dimension they occupy in space.

The relationships between the length of the sides and the size of the angles in a triangle can be used to solve many problems involving position, distance, angles and area.

1 B $0,4,0$, C $5,0,0$, D $(5,0,3)$

2 $(2.5,2,1.5)$, $(2.5,2,0)$

3 $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}\right)$

4 **a** $(2.5, 2, 1.5)$

b It is the same as the mid-point of [OA] which is what you would expect as the diagonals meet in the centre of the cuboid.

5 **a** $\text{BC} = \sqrt{4^2 + 5^2} = \sqrt{51}$

b $\text{BD} = \sqrt{\left(-\sqrt{41}\right)^2 + 3^2} = \sqrt{50}$

6 $\text{BC} = \sqrt{p^2 + q^2}$

7 $\text{BD} = \sqrt{r^2 + \sqrt{p^2 + q^2}^{\,2}} = \sqrt{p^2 + q^2 + r^2}$

8 $\text{BD} = \sqrt{4^2 + 5^2 + 3^2} = \sqrt{50}$

9 length = $= \sqrt{x_2 - x_1{}^2 + y_2 - y_1{}^2 + z_2 - z_1{}^2}$

Justification might be that a cuboid can be constructed between any two points in 3-dimensional space with lengths of sides equal to the difference between the corresponding pairs of coordinates. The result from question **6** can then be used to find the length.

Factual: What is the formula for finding the length of the line segment joining $x_1 + x_2, y_1 + y_2, z_1 + z_2$?

Answer: $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}\right)$

Factual: What is the formula for finding the distance between $x_1 + x_2, y_1 + y_2, z_1 + z_2$?

Answer: distance $= \sqrt{x_2 - x_1{}^2 + y_2 - y_1{}^2 + z_2 - z_1{}^2}$

Conceptual: How are the formulae for distance between 2 points and mid-point of a line segment in 3 dimensions related to the same formulae in 2 dimensions?

Answer: The properties of shapes depend on the dimension they occupy in space.

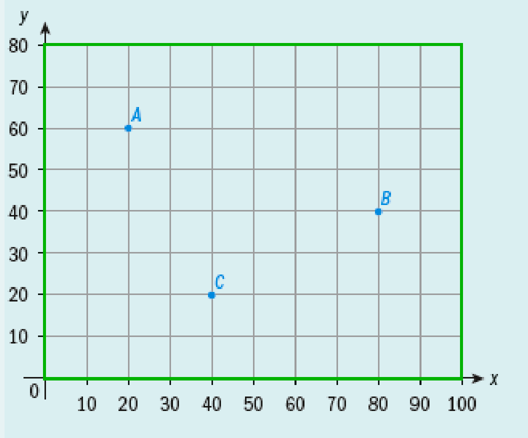
The relationships between the length of the sides and the size of the angles in a triangle can be used to solve many problems involving position, distance, angles and area.

Developing inquiry skills

Look back at the opening problem for the chapter. You were trying to divide the territorial waters between three islands. The positions of the islands can be modelled as shown:

The lines $x = 0$, $x = 100$, $y = 0$ and $y = 80$ mark the boundary of international waters and distances are given in kilometres. The islands are given exclusive fishing rights within these boundaries with the island closest to a point having the rights at that point.

How can you find the distances between each of the islands?



Answer: The distances between the islands can be found using Pythagoras' theorem, and are *AB* = 63.2 km, *AC* = 44.7 km, *BC*= 44.7 km

3.2 The equation of a straight line in 2 dimensions

Investigation 2

Conceptual understanding:

Equations of straight lines can be rearranged into equivalent equations, such as gradient−intercept form, general form, point−*gradient* form, all representing the gradient of the line and its position on the coordinate grid.

1 The gradient is 0.5 whichever points are chosen.

2 $m = \dfrac{y-3}{x-1}$ or $0.5 = \dfrac{y-3}{x-1}$

3 $m = \dfrac{y-y_1}{x-x_1}$

4 $m\ \ x - x_1\ \ = y - y_1 \ \Rightarrow y - y_1 = m\ \ x - x_1$

5 $y - 3 = 0.5\ \ x - 1$

6 $y = 0.5x + 2.5$

7 $x - 2y + 5 = 0$

Conceptual: What are the three forms of the equation that give the relation between the *x*-coordinates and the *y*-coordinates of all points on a straight line?

Answer: Equations of straight lines can be rearranged into equivalent equations, such as gradient−intercept form, general form, point−*gradient* form, all representing the gradient of the line and its position on the coordinate grid.

Investigation 3

Conceptual understanding:

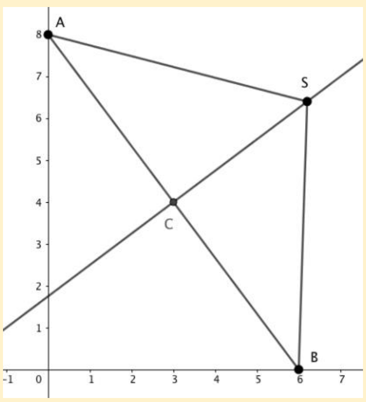
All points equidistant from two fixed points lie on the perpendicular bisector of the line segment joining those two points.

1 The perpendicular bisector has equation $y-4=\frac{3}{4}(x-3)$ or $y=\frac{3}{4}x+\frac{7}{4}$

2 The ship has coordinates $(7,7)$ and $7=\frac{3}{4}\times 7+\frac{7}{4}$

3 SA = SB = $\sqrt{50}=7.07$ km. The ship is equidistant from the two points.

4 From the diagram it can be seen that the triangles SAC and SBC have one side in common (SC), and AC = BC as C lies on the perpendicular bisector. Both angles at C must be 90° as it is perpendicular to [AB] and hence we can use Pythagoras to show the lengths SA and SB must be equal for any point on the perpendicular bisector.



Conceptual: What is the relation between all points on the perpendicular bisector of the line segment joining two points?

Answer: All points equidistant from two fixed points lie on the perpendicular bisector of the line segment joining those two points.

Developing inquiry skills

Look back at the opening problem for the chapter. You were trying to divide the territorial waters between three islands.

1 Find the equations of the perpendicular bisectors between the islands.

2 Find the coordinates of the point where the perpendicular bisectors between A and B and A and C meet.

Verify that the perpendicular bisector between B and C also passes through this point. Do you think these lines divide the waters in a fair way? Justify your answer.

Answer: The midpoint between A and B is $50, 50$, A and C is $30, 40$ and B and C is $60, 30$.

Equations of perpendicular bisectors are, $y=-2x+150$ and $y=3x-100$ (or $-x+2y=50$, $2x+y=150$, $-3x+y=-100$)

Bisectors meet at $50, 50$. This point also lies on the perpendicular bisector of BC, as can be verified by substituting.

In one sense it is fair as everyone has access to the water closest to them. It could be argued that it is not fair, as not all islands get the same area. This should only be surmised 'by eye' at this stage.

TOK

Should we use gradient or slope? $y = mx + b$ or $y = mx + c$ or $ax + by + c = 0$? Does the language used hinder understanding? How can we deal with this issue?

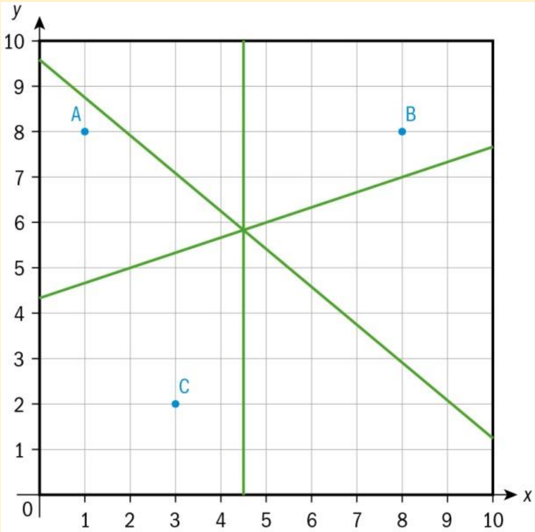
3.3 Voronoi diagrams

Investigation 4

Conceptual understanding:

The edges of a cell on a Voronoi diagram lie along the perpendicular bisectors of the line segments joining the points.

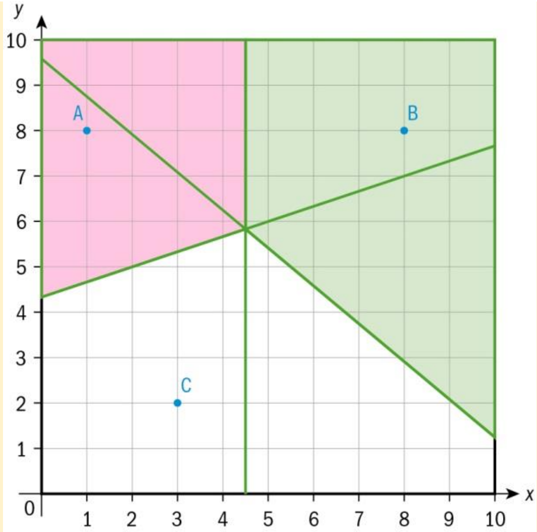
1



2 $x = 4.5$

3 Lines shown on the diagram above.

4 It might not be clear which areas need to be shaded so this could lead to a discussion on which strategies students used to decide.



Conceptual: What can be said about the boundaries of a Voronoi diagram?

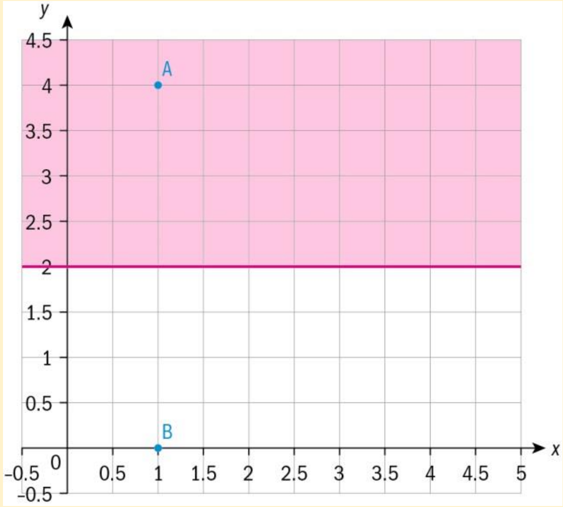
Answer: The edges of a cell on a Voronoi diagram lie along the perpendicular bisectors of the line segments joining the points.

Investigation 5

Conceptual understanding:

The incremental algorithm allows the edges of a Voronoi diagram to be easily found.

1

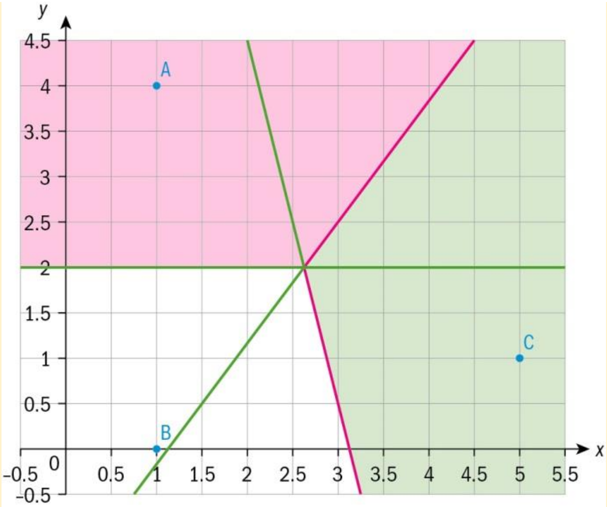


2 Perpendicular bisector of AC**:** $-4x + 3y = -4.5$

Perpendicular bisector of BC: $4x + y = 12.5$

They all meet in a single point.

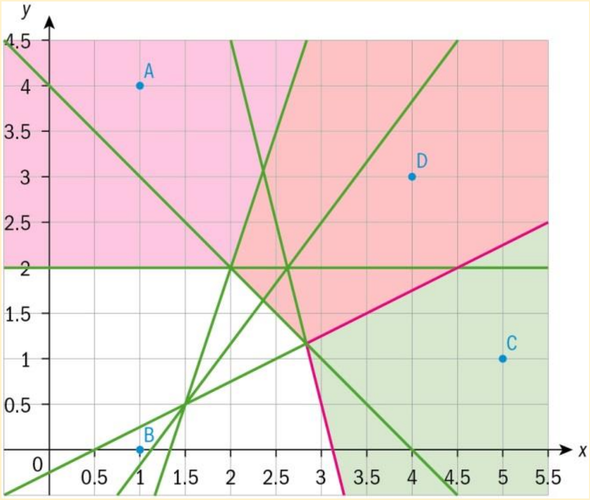
3 and **4**



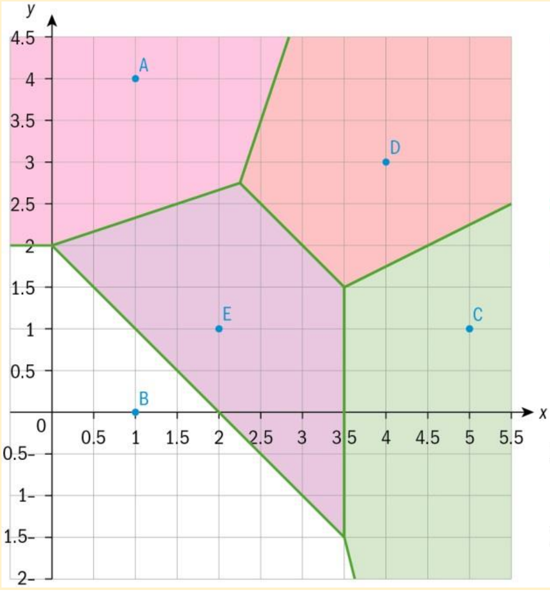
5 Perpendicular bisector of AD: $-3x + y = -4$

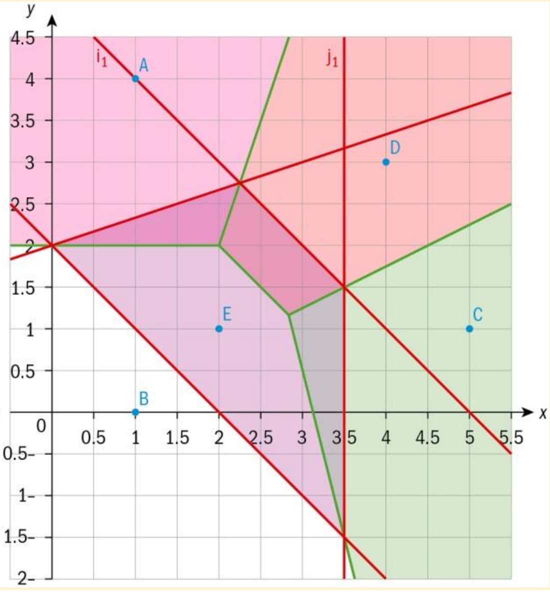
Perpendicular bisector of BD: $x + y = 4$

Perpendicular bisector of CD: $x - 2y = 0.5$



6





7 Adding D to the original diagram would have introduced another 3 perpendicular bisectors to an already crowded diagram and so it would have been difficult to construct the diagram directly.

8 **Conceptual:** Why is the incremental algorithm used in the construction of a Voronoi diagram?

Answer: The incremental algorithm allows the edges of a Voronoi diagram to be easily found.

Investigation 6

Conceptual understanding:

The solution to the toxic waste dump problem lies at a vertex of the Voronoi diagram.

1 **d** The largest circle will pass through three points.

2 **a** Three edges means there are three regions incident on the vertex. Let the sites in these regions be A, B and C and the vertex be V. If VA $= x$ then VB $= x$ as V lies on the perpendicular bisector of A and B, similarly VC $= x$, and so all three sites are equidistant from V.

b The distance to each of the points would equal the radius of the circle.

3 If a point D was such that VD $< x$ then V would be closer to D than to A. This is a contradiction to the statement that V is on the boundary of the cell containing A.

4 **Conceptual:** Explain how the construction of a Voronoi diagram can lead to a solution of the toxic waste dump problem.

Answer: The solution to the toxic waste dump problem lies at a vertex of the Voronoi diagram.

Developing inquiry skills

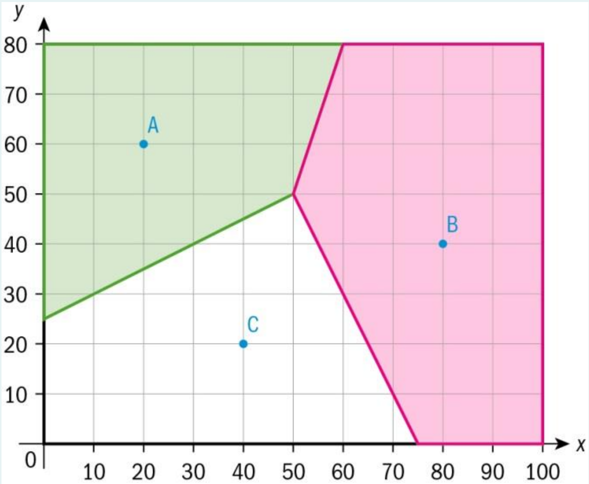
Look back at the opening problem for the chapter. You were trying to divide the territorial waters between three islands.

1 Draw the Voronoi diagram showing the regions in which each of the three islands have exclusive fishing rights.

2 Find the area of each of these regions.

Answer:

1



2 Voronoi edges meet the boundaries of the diagram at $0, 25$, $75, 0$ and $60, 80$.

Each area is composed of a triangle and a trapezium.

A $\frac{1}{2} \times 50 \times 25 + \frac{1}{2} \times 30 \times \ 60 + 50 \ = 2275\ \text{km}^2$

B 3225 km^2

C $8000 - 3225 - 2275 = 2500\ \text{km}^2$

3.4 Displacement vectors

Investigation 7

Conceptual understanding:

Adding vectors gives the resultant effect of all the vectors acting successively.

1 **a** $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$

b−f The student should find that, whichever intermediate points they take, the sum is always $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$.

2 **a** and **b** Students' answers will vary.

c The sum will always be $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$.

d The vector sum will always be equal to the direct vector from the start to the finish point.

Conceptual: What is the geometric meaning of a vector sum?

Answer: Adding vectors gives the resultant effect of all the vectors acting successively.

TOK

A class discussion might include vectors are used to solve many problems in position location. This can be to save a lost sailor or destroy a building with a laser guided bomb. Are either or both acceptable?

TOK

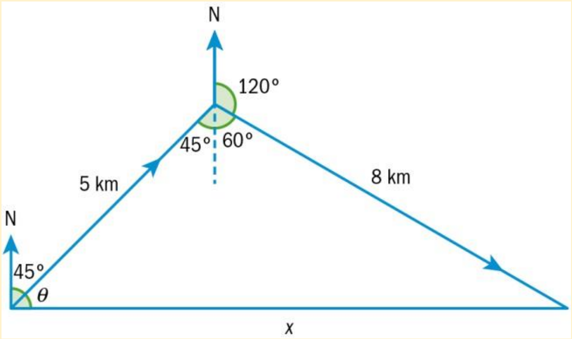
In how many ways can you prove Pythagoras' theorem?

Investigation 8

Conceptual understanding:

Vectors can be given as a magnitude and direction or written in component form, and using component form allows you to find the resultant easily.

1 **a**



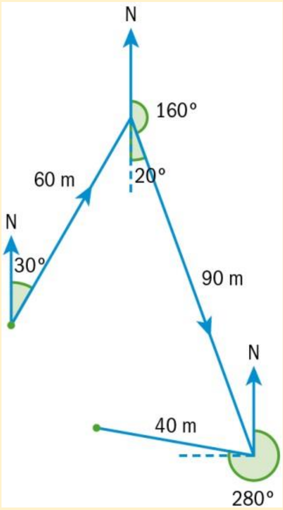
b $x^2 = 5^2 + 8^2 - 2\times 5\times 8\cos 105^0$ $x = 10.5$ km

$\frac{\sin\theta}{8} = \frac{\sin 105}{10.5}$ $\theta = 47.5^0$ Bearing is 45+47.5 = 092.5°

c **i** $\begin{pmatrix}3.54\\3.54\end{pmatrix}$, $\begin{pmatrix}6.93\\-4.0\end{pmatrix}$ **ii** 10.4 km east, −0.464 km north

iii $\begin{pmatrix}10.4\\-0.464\end{pmatrix}$ **iv** $\sqrt{10.4^2 + 0.464^2} = 10.5$ km, 90°+2.5°=092.5°

2 **a**



$\begin{pmatrix}30\\52\end{pmatrix} + \begin{pmatrix}30.8\\-84.6\end{pmatrix} + \begin{pmatrix}-39.4\\5.2\end{pmatrix}$ = $\begin{pmatrix}21.4\\-27.4\end{pmatrix}$

Distance = 34.8 m

b It is not intended that students do this question using sine and cosine rules just to appreciate that as the number of vectors increases it very quickly becomes more difficult to just use the lengths of the lines and the angles. Some might realize that they can create right-angled triangles which is an equivalent method to find the components.

Factual: How can you represent vectors?

Answer: Vectors can be represented by giving their magnitude and direction or using component form.

Conceptual: Which representation is easier to use to find a resultant?

Answer: Vectors can be given as a magnitude and direction or written in component form, and using component form allows you to find the resultant easily.

3.5 The scalar and vector product

Investigation 9

Conceptual understanding:

The scalar product can be used to find the angle between two vectors.

1 **a** $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$

b $|\mathbf{a}|^2 = 1+4+9 = 14$ $\quad |\mathbf{b}|^2 = 4+25+4 = 33 \quad AB^2 = 1+9+25 = 35$

c $\frac{1}{2}(14+33-35) = 6$

d 6

e The result is the same as that evaluated in part **d**.

2 Given $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ then $|\mathbf{a}|^2 = a_1^2 + a_2^2$ and $|\mathbf{b}|^2 = b_1^2 + b_2^2$

Because AB is $|\mathbf{b} - \mathbf{a}|$ then $AB^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2$

$$\frac{1}{2}\left(|\mathbf{a}|^2 + |\mathbf{b}|^2 - AB^2\right) = \frac{1}{2}\left((a_1^2 + a_2^2) + (b_1^2 + b_2^2) - \left((b_1 - a_1)^2 + (b_2 - a_2)^2\right)\right)$$

When the right hand side is expanded you get

$$\frac{1}{2}\left(a_1^2 + a_2^2 + b_1^2 + b_2^2 - (b_1^2 - 2a_1b_1 + a_1^2 + b_2^2 - 2a_2b_2 + a_2^2)\right)$$

Cancelling gives $a_1b_1 + a_2b_2 = \mathbf{a} \cdot \mathbf{b}$

3 A similar method for a three-dimensional vector gives $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

4 **a** The length AB can be found using the cosine rule.

$AB^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$ and rearranged to give the required form

$$|\mathbf{a}||\mathbf{b}|\cos\theta = \frac{1}{2}\left(|\mathbf{a}|^2 + |\mathbf{b}|^2 - AB^2\right)$$

b $\cos\theta = \dfrac{a_1b_1 + a_2b_2}{|\mathbf{a}||\mathbf{b}|} \quad \cos\theta = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{|\mathbf{a}||\mathbf{b}|}$

5 **i** $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix} = -2 + 6 = 4$

$|\mathbf{a}| = \sqrt{1^2 + 2^2} = \sqrt{5}$ and $|\mathbf{b}| = \sqrt{2^2 + 3^2} = \sqrt{13}$

$$\cos\theta = \frac{4}{\sqrt{5}\sqrt{13}} \text{ so } \theta = 60.3^{o}$$

ii $$\begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix} = -6 - 8 = -14$$

$$\cos\theta = \frac{-14}{\sqrt{20}\sqrt{17}} \text{ so } \theta = 139.4^{o}$$

6 **a** 0 **b** The vectors are perpendicular.

7 **Conceptual:** What can the scalar product be used to find?

Answer: The scalar product can be used to find the angle between two vectors.

TOK

Here is an explanation from Peter Webb.

All laws for vector arithmetic must have a specific property, being that the answer to any vector arithmetic question must be independent of how you lay out the axis.

You can add two vectors by putting the tail of one on the head of the other and forming a triangle. I am sure you have done this many times. You don't need to lay out a set of axis to do this. It's purely geometric. If you want to do this algebraically, you can lay down some *x*-*y* axes and calculate (a, b) + (c, d) = (a + c, b + d). But it doesn't matter how you lay down the axes, you get different numbers, but the resulting vector (the sum of the two vectors) is always the same.

There is an excellent reason why vector arithmetic has to be independent of the axis you select. Nature doesn't have axes, they are a human invention, so things in physics have to be independent of coordinate systems. And physics is believed to be isotropic − rotating an experiment should not affect an outcome, but it does affect the coordinate axes, so the layout of the coordinate system can't matter.

If you follow these rules, you have two choices about how vector multiplication in 2D space works. You can define multiplication as being a dot product which produces a scalar (a number), or the cross product which gives another vector perpendicular to the plane.

If you want vector arithmetic to model real physics, then the rules have to be independent of the coordinate system, and the dot and cross product are the only versions of multiplication that work, i.e. are independent of the (arbitrary) choice of axes. They are the only ones which are independent of the axes and hence can model real physics.

Investigation 10

Conceptual understanding:

The vector product of two vectors represents a vector perpendicular to the two vectors.

1 $$\begin{pmatrix} -5 \\ 7 \\ -4 \end{pmatrix}$$

2 In each case the vector product is 0, which means it is perpendicular to each of the two vectors.

3 The verification will also give 0.

Verifying part **4** will prove that the conjecture is always true.

4 The scalar product $\begin{pmatrix} a_2b_3 - b_2a_3 \\ -\ a_1b_3 - b_1a_3 \\ a_1b_2 - b_1a_2 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$

5 **Conceptual:** What can you say about the direction of the vector product of two vectors?

Answer: The vector product of two vectors represents a vector perpendicular to the two vectors.

International-mindedness

Bigha – India

Mou – China

Feddan – Egypt

Rai – Thailand

Tsubo – Japan

3.6 Vector equations of lines

TOK

Why are the symbolic representations of 3D shapes easier to work with than the actual drawings?

What does this tell us about our knowledge of mathematics in other dimensions?

Investigation 11

Conceptual understanding:

The equation of a line $\mathbf{r} = \mathbf{a} + \mathbf{b}t$ is defined by the vectors **a** and **b** where **a** is the position vector of a point on the line and **b** is a vector in the direction of the line.

1 $\overrightarrow{OC} = \overrightarrow{OA} + 2\overrightarrow{AB}$

2 $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 0\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \overrightarrow{OB} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \overrightarrow{OD} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 1\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

3 Any point P on the line can be written as $\overrightarrow{OP} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for some value of t.

4 The vector from A to any point not on the line will not be parallel to $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and hence cannot be written in this form.

5 Values of s are: A $s = 1$, B $s = 0$, C $s = -1$, D $s = 2$

6 **Factual:** What is the vector equation of a line in symbols?

Answer: $\mathbf{r} = \mathbf{a} + \mathbf{b}t$

7 **Conceptual:** How do we express the vector equation of a line in word?

Answer: The equation of a line $\mathbf{r} = \mathbf{a} + \mathbf{b}t$ is defined by the vectors **a** and **b** where **a** is the position vector of a point on the line and **b** is a vector in the direction of the line.

TOK

A good class discussion as students might be more confident with previously encountered work.

Investigation 12

Conceptual understanding:

The motion of a particle travelling with constant velocity can be given as the equation of a straight line with direction vector **v** and parameter *t* which represents the time from the initial position.

1 a After one hour the boat will be at point B with position vector
$\begin{pmatrix} 5 \\ 1 \end{pmatrix} + 1\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

b After two hours it will be at point C with position vector
$\begin{pmatrix} 5 \\ 1 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

c $r = \begin{pmatrix} 5 \\ 1 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

2 a x-coordinate will be 0 b $t = 2.5$

3 a x is negative, y is positive and $x = -y$ b $5 - 2t = -1 - t \Rightarrow t = 6$

4 **Conceptual:** How can the motion of a particle travelling with constant velocity be expressed and what do the variables stand for in this representation?

Answer: The motion of a particle travelling with constant velocity can be given as the equation of a straight line with direction vector **v** and parameter *t* which represents the time from the initial position.

Reflect: How might you find the least distance between two objects each moving with constant velocity?

How might you find the time when the bearing of one object to another, is in a given direction?

Answer: The relative position of B from A can be used to find the distance between A and B and the direction they are relative to each other.

Developing inquiry skills

Look back at the opening problem for the chapter. You were trying to divide the territorial waters between three islands.

Island A feels it is not getting a fair allocation of the area. An alternative is proposed whereby instead of the previous area it can have exclusive fishing rights for all of the region within 35 km of the centre of the island, including the international waters, except where this would overlap with an area closer to one of the other islands.

Vector methods will be used to find the area of this region.

a Use the diagram to write down the **vector** equations of the three perpendicular bisectors from the opening problem.

Let P and Q be the points on the perpendicular bisectors of [AB] and [AC] which are 35 km from A and on the edges of the Voronoi diagram.

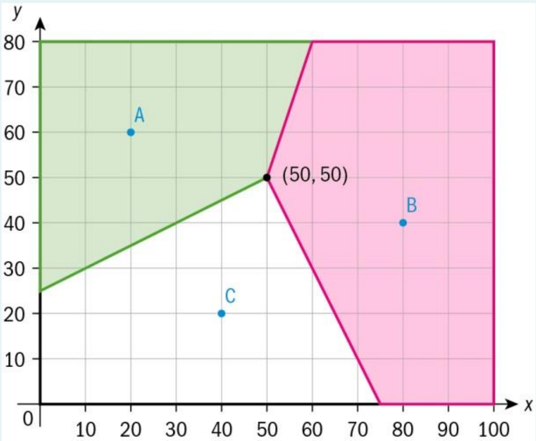
b Find the vectors $\overrightarrow{AP}$ and $\overrightarrow{AQ}$.

c Show the new region in which A has exclusive fishing rights on the diagram.

d Use the scalar product to find angle QAP.

e Find the area of the region in which island A has exclusive fishing rights.

Answer: This is a tricky problem so should be regarded as an extension question for the able students.

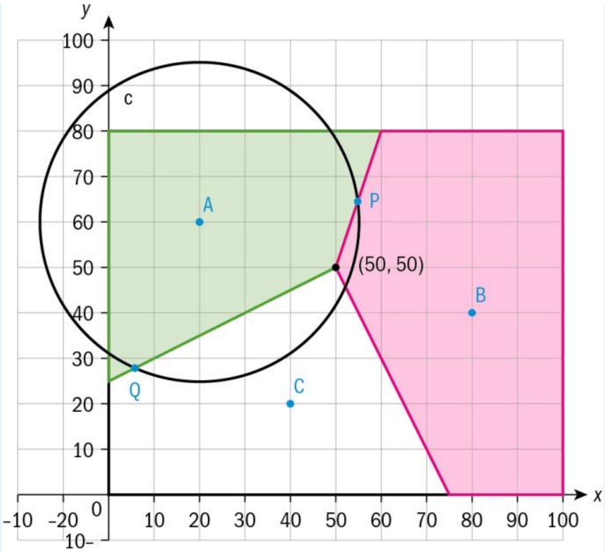


Let point (50,50) be M

a $r = \begin{pmatrix}50\\50\end{pmatrix} + t\begin{pmatrix}1\\3\end{pmatrix}$, $r = \begin{pmatrix}50\\50\end{pmatrix} + t\begin{pmatrix}2\\1\end{pmatrix}$ and $r = \begin{pmatrix}50\\50\end{pmatrix} + t\begin{pmatrix}-1\\2\end{pmatrix}$

(other forms are possible, in particular students might use the midpoints as the fixed point in the equation)

b



$$\boldsymbol{r} = \begin{pmatrix}50\\50\end{pmatrix} + t\begin{pmatrix}1\\3\end{pmatrix}\ ,\ \boldsymbol{r} = \begin{pmatrix}50\\50\end{pmatrix} + t\begin{pmatrix}2\\1\end{pmatrix}$$

Let the two points be P and Q

$$\overrightarrow{\mathrm{AP}} = \begin{pmatrix}50+t\\50+3t\end{pmatrix} - \begin{pmatrix}20\\60\end{pmatrix} = \begin{pmatrix}30+t\\-10+3t\end{pmatrix} \text{ so AP} = 35 = \sqrt{(30+t)^2 + (-10+3t)^2}$$

$t = 4.74...$

Note: there are 2 solutions but only one is on the edge of the Voronoi diagram.

$$\overrightarrow{\mathrm{AP}} = \begin{pmatrix}34.74\\4.23\end{pmatrix} \text{ and } \overrightarrow{\mathrm{AQ}} = \begin{pmatrix}50+2t\\50+t\end{pmatrix} - \begin{pmatrix}20\\60\end{pmatrix} = \begin{pmatrix}30+2t\\-10+t\end{pmatrix}$$

$$\text{AQ} = 35 = \sqrt{(30+2t)^2 + (-10+t)^2} \quad \text{so } t = -22.04...$$

Note: there are 2 solutions but only one is on the edge of the Voronoi diagram.

$$\overrightarrow{\mathrm{AQ}} = \begin{pmatrix}-14.08\\-32.04\end{pmatrix}$$

c The region bounded by the line Q to (50.50), from (50,50) to P and the major arc of the circle.

d Angle between $\overrightarrow{\mathrm{AP}}$ and $\overrightarrow{\mathrm{AQ}}$

$$\arccos\left(\frac{-615.5}{\sqrt{1224}\sqrt{1224}}\right) = 120.2° \text{ or 2.10 radians}$$

e Area of sector = $\frac{1}{2} \times 1224 \times (2\pi - 2.10) = 2560$

Use of vector product to find area of two triangles: $\overrightarrow{\mathrm{AM}} = \begin{pmatrix}30\\-10\end{pmatrix}$

$$\frac{1}{2}\left|\overrightarrow{AQ}\times\overrightarrow{AM}\right| = \begin{pmatrix}-14.08\\-32.04\end{pmatrix}\times\begin{pmatrix}30\\-10\end{pmatrix} = \frac{1}{2}\times\sqrt{94370} = 485.7$$

$$\frac{1}{2}\left|\overrightarrow{AP}\times\overrightarrow{AM}\right| = \begin{pmatrix}34.74\\4.23\end{pmatrix}\times\begin{pmatrix}30\\-10\end{pmatrix} = \frac{1}{2}\times\sqrt{136790} = 184.9$$

Total area $\approx$ 3230 km^2

Real-life Voronoi

Approaches to Learning: Thinking Skills: Evaluate, Critiquing, Applying

Exploration Criteria: Presentation (A), Personal engagement, (C) Use of mathematics (E)

IB Topic: Voronoi Diagrams

In this task students will be asked to create their own problem that can be answered using Voronoi diagrams. They are asked to take inspiration from the world around them, perhaps linking it to one of their subjects, their services or activities, their hobbies or interests. Students will need to clearly state the aim and context for their exploration. The rationale will help to demonstrate the personal engagement (Criterion C) of looking at a topic that is obviously relevant to a student.

Voronoi diagrams

In the examples, investigations and exercises in the chapter students have seen some of the potential uses of Voronoi diagrams and the types of problems they can solve.

Distance:

From the chapter students may consider the three farmhouses that are to be powered by wind where a new wind turbine is to be placed at the point equidistant from the 3 farms or 'the Toxic waste problem'.

Area:

From the chapter students could consider the three schools problem where children go to the school that is closest to their home when measured by a direct line and an estate agent wishes to construct a diagram, which shows in which school's catchment area a house lies.

Questions from the chapter can provide inspiration when students are creating their own questions.

Exploration

The aim is essential as it clearly states what students are going to complete in their exploration and a rationale will provide a context for why they have chosen a particular area of study. Choosing something that they are genuinely interested in studying and finding out more about is likely to make the exploration more successful for students.

Listing potentially interesting areas will be very useful when students actually have to decide what to do for their exploration using any mathematics topic. Encourage students to keep this list safe for use later.

Possible areas of interest could be, for example:

- School subjects you are most passionate about
- Particular areas you have studied in your subjects that have inspired you
- The service(s) you are involved in
- Activities you do in school and outside school
- Hobbies, games, sports you play
- Areas of interest you read about or watch regularly
- Careers or university courses you are interested in entering.

If needed, you could hint at the above suggestions.

For example:

A student who is involved in an animal conservation-based service could choose to model and analyse the territories of a chosen animal.

Beekeepers may use Voronoi diagrams because knowing the location of other beehives can be useful to prevent overcrowding as each beehive needs sufficient territory to survive.

Or

A student who is considering studying business when they leave school could have an interest in retail as so could choose to analyse the placement of a new shop in their hometown.

Other examples are:

- Urban planning, anthropology/history, cartography, urban settlement patterns, ecology, chemistry (crystal growth)

When writing their aim and context for their choice, make sure that students think about the region in which the Voronoi diagram is going to be drawn should be defined.

You do not want a problem with too many points!

Even if students do not go on to solve the problem, the process of thinking about what they could study and what needs to be considered when writing an aim and rationale is a very useful exercise.

Extension

This will require students to do a little research as it is not covered in the chapter. However, it is a useful application that may be more suitable for the type of problem that students have devised for themselves.

4 Modelling constant rates of change: linear functions and regressions

Essential understandings

Models are depictions of real-life events using expressions, equations or graphs while a function is defined as a relation or expression involving one or more variables. Creating different representations of functions to model the relationships between variables, visually and symbolically as graphs, equations and tables represents different ways to communicate mathematical ideas.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Different representations of functions, symbolically and visually as graphs, equations and tables provide different ways to communicate mathematical relationships.
- The parameters in a function or equation correspond to geometrical features of a graph and can represent physical quantities in spatial dimensions.
- Moving between different forms to represent functions allows for deeper understanding and provides different approaches to problem solving.
- Our spatial frame of reference affects the visible part of a function and by changing this "window" can show more or less of the function to best suit our needs.
- Changing the parameters of a trigonometric function changes the position, orientation and shape of the corresponding graph.
- Different representations facilitate modelling and interpretation of physical, social, economic and mathematical phenomena, which support solving real-life problems.
- Technology plays a key role in allowing humans to represent the real world as a model and to quantify the appropriateness of the model.
- Extending results from a specific case to a general form and making connections between related functions allows us to better understand physical phenomena.
- Generalization provides an insight into variation and allows us to access ideas such as half-life and scaling logarithmically to adapt theoretical models and solve complex real-life problems.
- Considering the reasonableness and validity of results helps us to make informed, unbiased decisions.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	Investigation
The vertical line test (on a graph) helps to identify types of relations that represent functions which map each input with exactly one output.	Investigation 1
To identify that a relation is a function from a graph, table, mapping diagram or context, write down any similar inquiry questions you could ask and investigate for another business or changing structure. The vertical line test (on a graph) helps to identify types of relations that represent functions which map each input with exactly one output.	Investigation 2
Function notation allows communication of corresponding input and output in expressions and equations.	Investigation 3

The domain includes all the possible input values (independent variables) for a function, and the range includes all the actual output values (dependent variables) and when modeling real world context, a suitable domain and associated range must be taken into consideration.	Investigation 4
The gradient of a linear function represents a constant rate of change between the dependent and independent variables, and the dependent variable's intercept represents the initial value of the function.	Investigation 5
Direct variation describes a relationship of direct proportion between two variables, and can be represented as a linear graph or equation with zero y-intercept.	Investigation 6
A piecewise linear function can be used to model contexts with different constant rates of change on two or more domains.	Investigation 7
The inverse of a linear function maps the function's output to the input and its domain to the range.	Investigation 8
The inverse of a linear function can be determined algebraically, by exchanging x and y in the equation, or graphically, by creating a mirror image about the line y = x.	Investigation 9
Two functions represent inverses of each other if their composition in either order produces the identity function.	Investigation 10
The nth term for an arithmetic sequence reflects a multiple of the common difference that will always be one less than the term as the sequence starts with a first term.	Investigation 11
The formula for the sum of an arithmetic sequence utilizes the symmetry in the sequence by adding the sequence to itself and compensating by halving the result.	Investigation 12
The parameters in a linear regression (least squares regression line) reflect values that minimise the sum of square residuals, thus minimising the error between the actual y-values and the y-values predicted by the line.	Investigation 13
Prediction from a linear regression may be more appropriate when the data displays a clear linear trend.	Investigation 14
Prediction from a linear regression may be appropriate when the data displays a clear linear trend and this can be interpreted as a minimum of moderate strength of the Pearson's product moment correlation coefficient.	Investigation 15

Syllabus sections covered in this chapter:

- SL2.2*
- SL2.3*
- SL2.4*
- SL2.5
- SL1.2
- SL1.8
- SL2.6
- SL4.4*
- SL4.10
- AHL2.7
- AHL2.9
- AHL4.13

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 140: Modelling constant rates of change: linear functions and regressions	Page 156: Example 4 Page 161: Example 6 Page 172: Example 10 Page 186: Example 19	Page 148: Example 2 Page 156: Example 4 Page 162: Example 7 Page 166: Example 8 Page 179: Example 13 Page 180: Example 14 Page 185: Example 18 Page 192: Example 20	Pages 153, 167, 177, 189, 195

Assessment opportunities

End of chapter summary	Chapter review	Exam-style questions
Page 197	Page 198	Page 200

Developing inquiry skills

Write down any similar inquiry questions you could ask and investigate for another business or charging structure. What information would you need to find?

Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Answer: Write down any similar inquiry questions you could ask and investigate for another business or changing structure. What information would you need to find? Student answers will vary.

4.1 Functions

Investigation 1

Conceptual understanding:

The vertical line test (on a graph) helps to identify types of relations that represent functions which map each input with exactly one output.

1 Run 1: C, Run 2: A, Run 3: G, Run 4: E, Run 5: B, Run 6: F

2 D does not fit: cannot be in two places at the same time.

Investigation 2

Conceptual understandings:

To identify that a relation is a function from a graph, table, mapping diagram or context, write down any similar inquiry questions you could ask and investigate for another business or changing structure.

The vertical line test (on a graph) helps to identify types of relations that represent functions which map each input with exactly one output.

1 **a** A given foot length will correspond to a particular shoe size, but if you know someone’s shoe size, you don’t necessarily know their foot length.

b Each student will have one birthday, so will have only one answer for the function. But for the non-function, some students will have multiple siblings, so multiple possible answers.

c Each x-input in the function has one corresponding output, equal to the square of x. For the non-function (the circle), there are two y-values on the graph for each x-value.

d In the function table there are no repeated x-values, but there are repeated y-values. In the non-function there are repeated x-values with different y-values.

e In the function, each x-value has one arrow drawn to a y-value. In the non-function, some x-values have two arrows drawn to two different y-values.

2 Students do not need to formally define a function, but should be able to articulate that a function predictably matches one output to an input, while a a non-function has multiple-outputs for one or more inputs.

3 **Factual:** What types of relations are functions?

Answer: Functions are relations where each input has only one corresponding output.

4 **Factual:** What are the different ways that a function can be represented?

Answer: Functions can be represented by tables, graphs, equations and mapping diagrams.

5 **Conceptual:** How can you identify that a relation is a function from a graph, table, mapping diagram or context?

Answer: To identify that a relation is a function from a graph, table, mapping diagram or context, write down any similar inquiry questions you could ask and investigate for another business or changing structure.

Investigation 3

Conceptual understanding:

Function notation allows communication of corresponding input and output in expressions and equations.

1 y = 790; Amir is 790 m from home after 10 minutes.

2 $x \approx 6$ and $x \approx 18$. This means that Amir is 600 m from home at about 6 minutes and at about 18 minutes.

3 Approximately 950 m at approximately 16 minutes, so $f(16) \approx 950$ or similar. (Students can use the table to see that the distance at 15 minutes is 945 m, and then the graph to see that the maximum occurs shortly after that.)

4 Approximately 27 minutes, when distance from home is 0, so $f(27) \approx 0$ or similar. (Table shows 45 m at 25 minutes, so zero must occur after this.)

5 **Conceptual:** What useful information does function notation communicate?

Answer: Function notation allows communication of corresponding input and output in expressions and equations.

Investigation 4

Conceptual understanding:

The domain includes all the possible input values (independent variables) for a function, and the range includes all the actual output values (dependent variables) and when modelling real world context, a suitable domain and associated range must be taken into consideration.

1 **Factual:** How can you see from the graph of $g(x)$ that the domain is $x \in \mathbb{R}$?

Answer: The graph extends to the left and right indefinitely in both directions, reaching every real number value for x.

2 $\{x \mid 0 \le x \le 55.2\}$ or [0, 55.2], as the graph begins at $x = 0$ and ends just after $x = 55$. (Note that 55.2 is approximate.)

3 $\{y \mid 0 \le y \le 12\,000\}$ or [0, 12 000] because the y-coordinates of the graph include all numbers below approximately 12 000 m.

4 $\{y \mid 0 \le y \le 10\,000\}$ or [0, 10 000].

5 **Conceptual:** How does the real-world context of a function influence its reasonable domain and range?

Answer: The domain includes all the possible input values (independent variables) for a function, and the range includes all the actual output values (dependent variables) and when modeling real world context, a suitable domain and associated range must be taken into consideration.

4.2 Linear models

TOK

You might want to look at the difference between a proof and a model.

- Is a model personal knowledge?

Models often solve real-world problems with mathematics as opposed to just problems in pure mathematics.

A good place to look at the connection between interpolation and extrapolation in terms of reliability.

Investigation 5

Conceptual understanding:

The gradient of a linear function represents a constant rate of change between the dependent and independent variables, and the dependent variable's intercept represents the initial value of the function.

1 Run 1: 1300 m travelled in 10 minutes.

Run 2: 1860 m travelled in 10 minutes (student estimates may vary between 1825 m and 1875 m).

2 Run 1: 130 m per minute, Run 2: approximately 186 m per minute

3 Run 1: Yes – the graph's gradient is constant, and he is running on flat ground.

Run 2: No – the graph's gradient changes. He is running downhill and then on flat ground; he will be faster running downhill than on flat ground.

4 130 m per minute

5 He ran at a speed of 130 m per minute throughout the run.

6 $y = 130x$

7 The gradient is 130 and represents the rate of change or speed, 130 m per minute. The y-intercept is 0 and corresponds to Amir's starting distance.

8 **Factual:** What do the letters m and c stand for in $y = mx + c$.

Answer: m represents the gradient and c represents the y-intercept.

Conceptual: What does the constant rate of change represent in a linear function and what does the y-intercept represent in terms of the dependent and independent variables?

Answer: The gradient of a linear function represents a constant rate of change between the dependent and independent variables, and the dependent variable's intercept represents the initial value of the function.

TOK

Is it OK to only use one type of mathematical and/or national term?

Whose job is it to impart knowledge and understanding?

How has technology influenced the notation that we use?

You might want to research how many words that Eskimos have for snow.

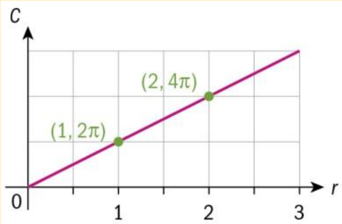
How do you think that this use of language affects understanding of space, time, colours, and objects?

Investigation 6

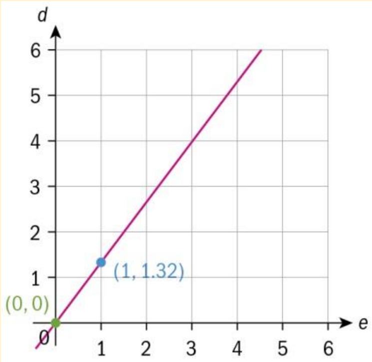
Conceptual understanding:

Direct variation describes a relationship of direct proportion between two variables, and can be represented as a linear graph or equation with zero y-intercept.

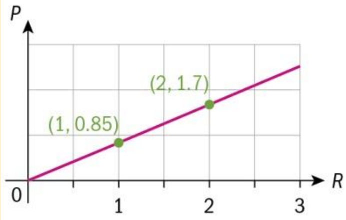
1 a i $C = 2\pi r$ **ii**



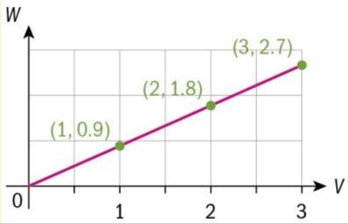
b i $e = 1.32d$ **ii**



c i If you pay 15% less for every item then you pay 85% of the price. This means that to calculate the price to be paid, you have to multiply the regular price by 0.85: $P = 0.85R$

ii 

d **i** For each dm^3 of this wood the weight increases by 0.9 kg. Hence: $W = 0.9V$

ii 

2 All the equations have the form $y = mx$, a linear function with zero y-intercept.

3 The graphs all have a y-intercept of 0 and a gradient of $m = \frac{y}{x}$, where (x, y) is a point on the line.

4 **Conceptual**: How can you identify a direct variation relationship from a graph, equation or context?

Answer: Direct variation describes a relationship of direct proportion between two variables, and can be represented as a linear graph or equation with zero y-intercept.

International-mindedness

The notation for functions was developed by a number of different mathematicians in the 17th and 18th centuries, you can ask students "how did the notation we use today become internationally accepted?"

Reflect: What does it mean to make a mathematical prediction?

Answer: To make a prediction for one variable, substitute a known value of the other variable.

Reflect: How do we use the modeling process to represent and make predictions about real-world contexts?

Answer: Identify the variables required to model the particular real-world context and follow the steps of the modelling process.

Investigation 7

Conceptual understanding:

A piecewise linear function can be used to model contexts with different constant rates of change on two or more domains.

1 The graph shows that the gradient is less steep in the middle portion of Amir's run, meaning that he ran more slowly on the rough terrain than before or after. He appears to run at the same speed before and after the rough patch, because the gradients appear equal.

2 Because there is not one constant rate of change – there are several.

3 You could model each of the three sections with a different linear function.

4

	Starting point	**Ending point**	**Gradient**	**Point–gradient equation**
Section 1	(0, 1120)	(4, 600)	$\frac{600-1120}{4-0} = -130$	y = −130x + 1120
Section 2	(4, 600)	(7, 390)	$\frac{390-600}{7-4} = -70$	y − 600 = −70(x − 4)
Section 3	(7, 390)	(10, 0)	$\frac{0-390}{10-7} = -130$	y − 390 = −130(x − 7)

5

	Equation	**Domain**
Section 1	y = −130x + 1120	$0 \le x < 4$
WSection 2	y = −70x + 880	$4 \le x < 7$
Section 3	y = −130x + 1300	$7 \le x \le 10$

6 Amir runs 130 meters per minute in the first section for 4 minutes, 70 meters per minute in the second section for 3 minutes, and 130 meters per minute for the last 3 minutes.

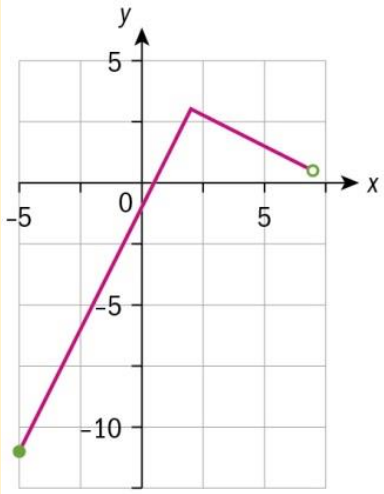
7 **a** Piece 3, because $x = 7$ lies in the domain $7 \le x \le 10$, not $4 \le x < 7$.

b If the domains overlap, the function may have more than one y-value for each x-value, and so will not be a function.

8 **a** $f(3) = -130(3) + 1120 = 730$. After 3 minutes, Amir is 730 m from home.

b $f(8) = -130(8) + 1300 = 260$. After 8 minutes, Amir is 250 m from home.

c Sketch the graph to see that $f(x) = 500$ is in the second piece:



Solving $500 = -70x + 880$ gives $x = 5.43$. Amir is 500 m from home after 5.43 minutes.

9 **a** $-130(4) + 1120 = 600$ and $-70(4) + 880 = 600$

b $-70(7) + 880 = 390$ and $-130(7) + 1300 = 390$

c Because the pieces are linear, and straight lines have no breaks or jumps.

10 **Factual**: What is a piecewise function and how do you evaluate a piecewise function for a specific value?

Answer: A piecewise function is a function that is defined by two or more different formulas on different, non-overlapping domains. To evaluate the function for a particular *x*-value, we look to see in which domain the *x*-value lies and substitute it in the corresponding formula.

11 **Factual**: How do you check that a piecewise function is continuous?

Answer: We substitute the endpoints of the domain where the pieces meet and check that the formulas give the same outputs.

12 **Conceptual**: When is a piecewise linear function a useful model?

Answer: A piecewise linear function can be used to model contexts with different constant rates of change on two or more domains.

Developing inquiry skills

Looking back at the opening problem, will the taxi driver reach his desired annual salary if he keeps his profit per kilometer at $2.50?

Answer: Let k represent the number of kilometers that the taximeter is on per day and P represent the taxi driver's daily profit. Then $P = 20 + 2.5k$, $10 \le k \le 40$. The maximum daily profit is $P(40) = \$120$. If the taxi driver earns this every day of the year, $\$120 \times 365 = £43\,800$, which is still less than $44 000. So the taxi driver must increase his profit per kilometer to reach his desired annual salary.

How can you model the taxi driver's **daily** profit for different values of profit per kilometer? What about **monthly** profit? State any assumptions that you make in your model.

Answer: When profit per kilometer is $2.50, then $P = 20 + 2.5k$.

When profit per kilometer is $2.80, then $P = 20 + 2.8k$

In general, if profit per kilometer is r, then daily profit is $P = 20 + rk, 10 \le k \le 40, r \ge 2.5$

The monthly profit M will depend on the number of days worked in a month, d:
$M = (20 + rk)d, 10 \le k \le 40, r \ge 2.5, 0 \le d \le 31$

For example, assuming he works 20 days per month, $M = (20 + rk) \times 20$

This model assumes, among other things, that the kilometers driven per day, k, will be constant for the month.

TOK

This is a good opportunity to debate, in teams or pairs. First, define mathematical rigour.

Ask opening questions such as: What can analysis offer us that graphing a function cannot?

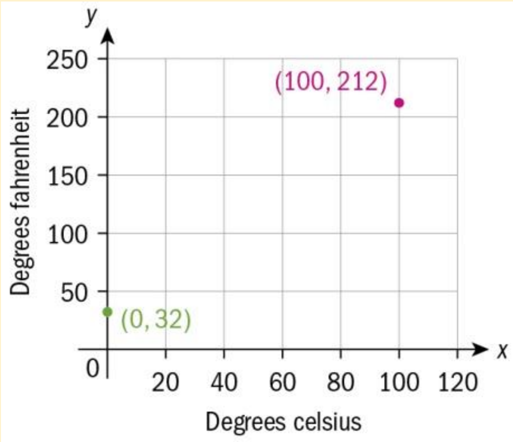
What are the strengths of a function?

4.3 Inverse functions

Investigation 8

Conceptual understanding:

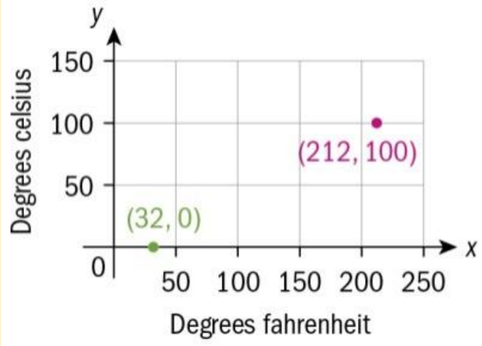
The inverse of a linear function maps the function's output to the input and domain to the range.

1 

2 $m = \dfrac{212 - 32}{100 - 0} = \dfrac{9}{5} = 1.8$, c = 32. So $F(x) = 1.8x + 32$

3 $F(37) = 1.8 \times 37 + 32 = 98.6$, so the function predicts correctly.

4 The independent and dependent variables switch because she is reversing the conversion: the input is now Fahrenheit and the output is Celsius.

5 

6 $m = \dfrac{100 - 0}{212 - 32} = \dfrac{5}{9} \approx 0.556$

7 This time the y-intercept is not one of the given points, so use point–gradient and convert to gradient–intercept form:

$$y - 0 = \frac{5}{9}(x - 32)$$
$$C(x) = \frac{5}{9}x - 17.8 \text{ (3 s.f.)}$$

To check, this model should predict a Celsius temperature of C = 37 for $x = 98.6$:

$C(98.6) = \dfrac{5}{9}(37) - 17.8 = 37.0 \text{ (3 s.f.)}$; it does.

Since we wish to convert Fahrenheit to Celsius, we input $x = -459.67$ into $C(x)$:

$C(-459.67) = \dfrac{5}{9}(-459.67) - 17.8 = -273 \text{ (3 s.f.)}$

8 a

	(x, C(x))	(x, F(x))
Freezing	(32, 0)	(0, 32)
Boiling	(212, 100)	(100, 212)
Body temperature	(98.6, 37)	(37, 98.6)
Absolute zero	(−459.76, −273)	(−273, −459.76)

b The point (a, b) on the graph of $C(x)$ corresponds to the point (b, a) on the graph of $F(x)$. This makes sense, because on the Celsius graph a is in Fahrenheit and b is in Celsius. On the Fahrenheit graph, their order is switched. Also, we are converting in the opposite direction.

c

	C(x)	F(x)
Domain	$\{x \mid x > -459.67\}$	$\{x \mid x > -273\}$
Range	$\{y \mid y > -273\}$	$\{y \mid y > -459.67\}$

d The domain of $C(x)$ corresponds to the range of $F(x)$ and vice versa. This makes sense, because the inputs of $C(x)$ are Fahrenheit temperatures, which correspond to the outputs of $F(x)$, and vice versa.

9 **Conceptual:** How are the inputs and outputs of inverse functions related? How are the practical domain and range of a linear model related to the domain and range of its inverse?

Answer: The inverse of a linear function maps the function's output to the input and its domain to the range.

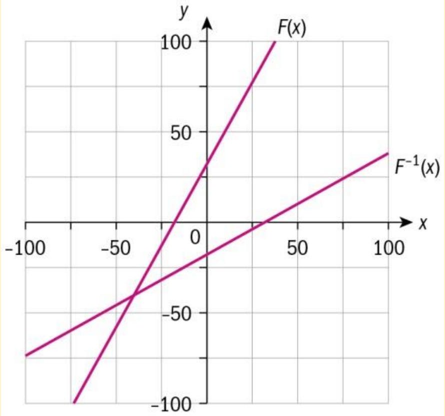
Investigation 9

Conceptual understanding:

The inverse of a linear function can be determined algebraically, by exchanging x and y in the equation, or graphically, by creating a mirror image about the identity line $y = x$.

1 Students own answers.

2 Graphs: F, F^{-1}



3 a $y = 3x$

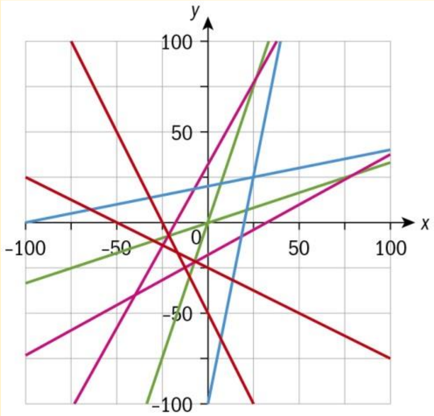
$$x = 3y$$
$$y = \frac{x}{3}$$
$$f^{-1}(x) = \frac{x}{3}$$

b $y = 5x - 100$

$$x = 5y - 100$$
$$x + 100 = 5y$$
$$y = \frac{x}{5} + 20$$
$$g^{-1}(x) = \frac{x}{5} + 20$$

c $y = -\frac{1}{2}x - 25$

$$x = -\frac{1}{2}y - 25$$
$$x + 25 = -\frac{1}{2}y$$
$$y = -2x - 50$$
$$h^{-1}(x) = -2x - 50$$



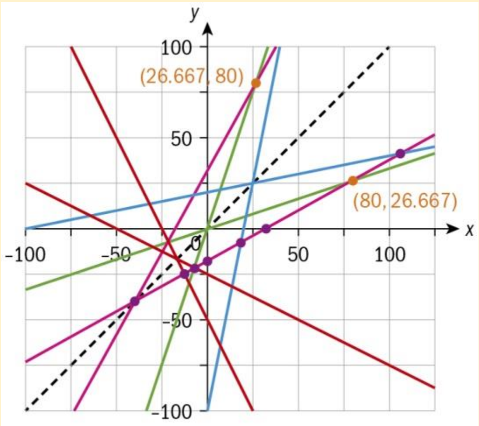
Graphs: f, f^{-1} in green; g, g^{-1} in purple; h, h^{-1} in orange

4 a

Function	y-intercept	x-intercept	Point of intersection with inverse
F(x)	(0, 32)	(−17.8, 0)	(−40, −40)
$F^{-1}(x)$	(0, −17.8)	(32, 0)	(−40, −40)
f(x)	(0, 0)	(0, 0)	(0, 0)
$f^{-1}(x)$	(0, 0)	(0, 0)	(0, 0)
g(x)	(0, −100)	(20, 0)	(25, 25)
$g^{-1}(x)$	(0, 20)	(−100, 0)	(25, 25)
h(x)	(0, −25)	(−50, 0)	(−16.7, −16.7)
$h^{-1}(x)$	(0, −50)	(−25, 0)	(−16.7, −16.7)

b The x-intercept of a function is the y-intercept of its inverse, and vice versa. The x and y-coordinates are switched. Also, the intersection point is always of the form (a, a).

5



The graphs are mirror images or symmetric across the line $y = x$. Individual points on the graphs are also mirror images: (26.7, 80) becomes (80, 26.7), and in general (a, b) becomes (b, a). We could also write this $f(26.7) = 80$ and $f^{-1}(80) = 26.7$.

6 **Conceptual:** How can the relationship between inverse functions be seen in their graphs and their equations?

Answer: The inverse of a linear function can be determined algebraically, by exchanging x and y in the equation, or graphically, by creating a mirror image about the identity line $y = x$.

TOK

What can you perceive from a graph with a labelled horizontal and/or vertical axis?

What about an unscaled axis?

Is it by intuition that we seek to have both axes the same scale?

How accurate is a visual representation of a mathematical concept?

Investigation 10

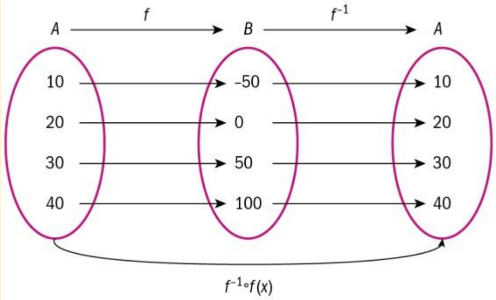
Conceptual understanding:

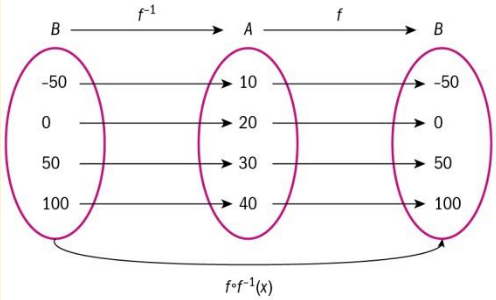
Two functions represent inverses of each other if their composition in either order produces the identity function.

1 **a** $f(10) = 5(10) - 100 = -50$ **b** $f^{-1}(-50) = -\frac{50}{5} + 20 = 10$

This example shows the general principle of an inverse, that $f(a) = b$ means that $f^{-1}(b) = a$. The inverse function 'undoes' the function.

2 **a**





b Both compositions return the original input, i.e. $f \circ f^{-1}(x) = x$ and vice versa.

3 The composition of inverses is the identity function. This makes sense because the inverse function undoes the action of the function.

4 Inverse: $x = 3y + 2$

$$y = \frac{x}{3} - \frac{2}{3}$$

$$f^{-1}(x) = \frac{x}{3} - \frac{2}{3}$$

$$\begin{aligned} f \circ f^{-1}(x) &= f\left(\frac{x}{3} - \frac{2}{3}\right) \\ &= 3\left(\frac{x}{3} - \frac{2}{3}\right) + 2 \\ &= x - 2 + 2 \\ &= x \end{aligned}$$

Similarly for the other direction.

5 **Conceptual:** How do you prove that two functions are inverses?

Answer: Two functions represent inverses of each other if their composition in either order produces the identity function.

4.4 Arithmetic sequences and series

TOK

Students could research the reason as to why we charge interest on a loan and compare this with the perspectives in other societies such as where money in Islam is not regarded as an asset from which it is ethically permissible to earn a direct return. The Qur'an (2:279) sees interest as inequitable, as implied by the word "zulm" in Arabic which translates as "oppression, exploitation, and the opposite of justice". There is no real loaning in Islam since lenders achieve ownership in the estates that they finance.

This allows students to view the perspectives of other societies and decide to what extent they agree with the charging of interest.

Investigation 11

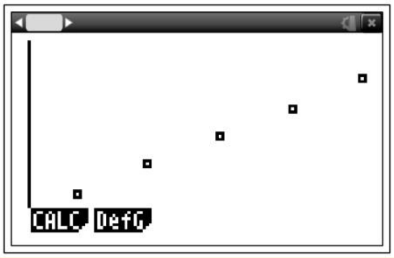
Conceptual understanding:

The *n*th term for an arithmetic sequence reflects a multiple of the common difference that will always be one less than the term as the sequence starts with a first term.

1

Years in job (term number)	1	2	3	4	5
Annual salary	39 000	39 900	40 800	41 700	42 600
Difference between terms	—	39 900 − 39 000 = 900	40 800 − 39 900 = 900	41 700 − 40 800 = 900	42 600 − 41 700 = 900

The salary is increasing at a constant rate.



2 a

The gradient is 900, which is the difference between consecutive terms.

b *y*-intercept =38 100. Since the line has gradient 900 and passes through the point (1, 39 000), the *y*-intercept at 0 will have a *y*-value 900 less than this, so the *y*-intercept is 38 100.

3 **Factual:** How do you determine whether a sequence is arithmetic without graphing it?

Answer: Look for a common difference between terms.

4 **Factual:** If a sequence is arithmetic, how are the parameters of the corresponding linear function related to the first term and difference between terms of the sequence?

Answer: If the corresponding linear function is $y = mx + c$, then m is the common difference and c is the first term minus the common difference.

5

Age	24	25	26	27	...	64	65
Number of term (n)	1	2	3	4	...	41	42
Number of differences added (d = $900)	0	1	2	3	...	40	41
Term (a_n)	39 000	39 000 + 900 = 39 900	39 000 + 2 × 900 = 40 800	39 000 + 3 × 900 = 41 700	...	39 000 + 40 × 900 = 75 000	39 000 + 41 × 900 = 75 900

6 **Conceptual:** How can you describe the *n*th term of an arithmetic sequence?

Answer: The *n*th term for an arithmetic sequence reflects a multiple of the common difference that will always be one less than the term as the sequence starts with a first term.

Investigation 12

Conceptual understanding:

The formula for the sum of an arithmetic sequence utilizes the symmetry in the sequence by adding the sequence to itself and compensating by halving the result.

1 The formula for the sum of an arithmetic sequence utilizes the symmetry in the sequence by adding the sequence to itself and compensating by halving the result.

2 1 + 100 = 101, 2 + 99 = 101, 3 + 98 = 101. They all sum to 101.

3 50 pairs, because there are 100 numbers. 50 × 101 = 5050

4 5 + 20 = 25, 8 + 17 = 25, 11 + 14 = 25

Sum = 25 × 3 = 75, which agrees with adding them in order.

5 **Factual:** Generalise your method to a formula for finding the sum of any arithmetic series.

Answer: $S_n = (a_1 + a_n) \times \frac{n}{2}$

6 $S_{42} = (39\,000 + 75\,900) \times \frac{42}{2} = \$2\,412\,900$

7 **Factual:** When is it more useful to represent a context as an arithmetic series than as a linear function?

Answer: When you wish to add up a sequence of values of the dependent variable, a series is a better representation to do this.

8 **Conceptual:** How is an arithmetic series represented and calculated?

Answer: The formula for the sum of an arithmetic sequence utilizes the symmetry in the sequence by adding the sequence to itself and compensating by halving the result.

TOK

Questions that can be used to stimulate a discussion or a blog post about the nature of mathematics include:

- What is mathematics?
- Is it all shared knowledge?
- How can you develop personal knowledge in mathematics?
- Do you just use formulas to reason and evaluate?
- Does faith ever have a place in mathematics?

Developing inquiry skills

Looking back at the opening problem, how can you model his daily and monthly profits if he increases the profit per kilometer by $0.30 each month?

Answer: As before, the number of km the taximeter is on per day is k, number of days worked per month is d. Let $P_n(k)$ represent the daily profit in month n. Then

$$P_1(k) = 20 + 2.5k$$

$$P_2(k) = 20 + 2.8k$$

$$P_3(k) = 20 + 3.1k$$

Notice that the profits per kilometer form an arithmetic sequence, so
$P_n(k) = 20 + (2.5 + (n-1)\times 0.3)k$

Then if the taxi driver has the taximeter on for k kilometers daily, works d days monthly, in month n the profit will be
$M_n(k) = [20 + (2.5 + (n-1)\times 0.3)k]\times d = (20 + 2.5k)d + (0.3kd)(n-1)$.

Will the taxi driver reach an annual salary of $44000 in a year's time if he increases his profit per kilometer by $0.30 each month? If not, how long will it take?

Answer: The monthly profits M_n M_n form an arithmetic sequence with first term $(20 + 2.5k)d$ and common difference $0.3kd$. To find the annual profit, find the sum of 12 monthly profits with an arithmetic series:

$$\sum_{i=1}^{12} M_i = \frac{12}{2}(2(20 + 2.5k)d + 11(0.3kd)) = 6d(40 + 8.3k)$$

Whether the driver makes his annual profit will depend on his average kilometers driven, k, and number of days driving per month, d. If he works 20 days per month driving the maximum 40 km daily, then his annual salary will be $6\cdot 20(40 + 8.3\cdot 40) = \$44640$, which exceeds $44 000.

4.5 Linear regression

Investigation 13

Conceptual understanding:

The parameters in a linear regression (least squares regression line) reflect values that minimise the sum of square residuals, thus minimising the error between the actual y-values and the y-values predicted by the line.

1 Answers will vary. Reasons may include that one line goes through 'the middle' of the data set, is closer to more of the points or will give better approximations for the data set. The purpose here is for students to begin thinking about how to measure goodness of fit.

2 **a** Since the residual is $y_i - f(x_i)$, a positive value indicates that the actual value is higher than the prediction, so $f(x)$ underestimates. A negative residual indicates that $f(x)$ is an overestimate.

b The line with better fit will have less error or distance to the data points overall, so its residuals should be generally smaller.

3	**x**	**y**	**Predicted y using L_1**	**Residual using**	**Predicted y using L_2**	**Residual using**

Point				L_1		L_2
(1, 3)	1	3	$0.81\times1+2.36=3.17$	$3-3.17=-0.17$	$0.65\times1+3.08=3.73$	$3-3.73=-0.73$
(2, 5)	2	5	$0.81\times2+2.36=3.98$	$5-3.98=1.02$	$0.65\times2+3.08=4.38$	$5-4.38=0.62$
(3, 3)	3	3	$0.81\times3+2.36=4.79$	$3-4.79=-1.79$	$0.65\times3+3.08=5.03$	$3-5.03=-2.03$
(4, 7)	4	7	$0.81\times4+2.36=5.60$	$7-5.60=1.40$	$0.65\times4+3.08=5.68$	$7-5.68=1.32$
(5, 5)	5	5	$0.81\times5+2.36=6.41$	$5-6.41=-1.41$	$0.65\times5+3.08=6.33$	$5-6.33=-1.33$
(6, 9)	6	9	$0.81\times6+2.36=7.22$	$9-7.22=1.78$	$0.65\times6+3.08=6.98$	$9-6.98=2.02$
(7, 7)	7	7	$0.81\times7+2.36=8.03$	$7-8.03=-1.03$	$0.65\times7+3.08=7.63$	$7-7.63=-0.63$
(8, 9)	8	9	$0.81\times8+2.36=8.84$	$9-8.84=0.16$	$0.65\times8+3.08=8.28$	$9-8.28=0.72$

4 The sum of the residuals for Line 1 is 0.04 and for Line 2 is −0.04. These are both very small, because the positive and negative residuals mainly cancelled.

5

Point	Residual using L_1	Square of residual using L_1	Residual using L_2	Square of residual using L_2
(1, 3)	−0.17	0.0289	−0.73	0.5329
(2, 5)	1.02	1.0404	0.62	0.3844
(3, 3)	−1.79	3.2041	−2.03	4.1209
(4, 7)	1.40	1.9600	1.32	1.7424
(5, 5)	−1.41	1.9881	−1.33	1.7689
(6, 9)	1.78	3.1684	2.02	4.0804
(7, 7)	−1.03	1.0609	−0.63	0.3969
(8, 9)	0.16	0.0256	0.72	0.5184
		SS_{res} = 12.48		SS_{res} = 13.55

6 Line 1 has the smaller sum of squared residuals (12.48 < 13.55), so is the better fit.

7 **a** The squares become larger in area as the line becomes a worse fit, and smaller as the line becomes a better fit.

b The area of each square is a sum of square residuals, so larger areas indicate more error and a worse fit.

c The gradient and the y-intercept can be changed.

8 **Conceptual:** How do we use sum of squares residuals to define the parameters in the least squares regression line and why does this lead to a line of best fit?

Answer: The parameters in a linear regression (least squares regression line) reflect values that minimise the sum of square residuals, thus minimising the error between the actual y-values and the y-values predicted by the line.

TOK

Questions that a teacher might like to pose include:

- Is it a bad idea to try to solve problems using our "gut feeling", intuition, or use reason and evidence to make a decision?
- Where does that gut feeling come from?
- When has it helped you?
- When has it misled you?

International-mindedness

The correlation between smoking and lung cancer was "discovered" using mathematics.

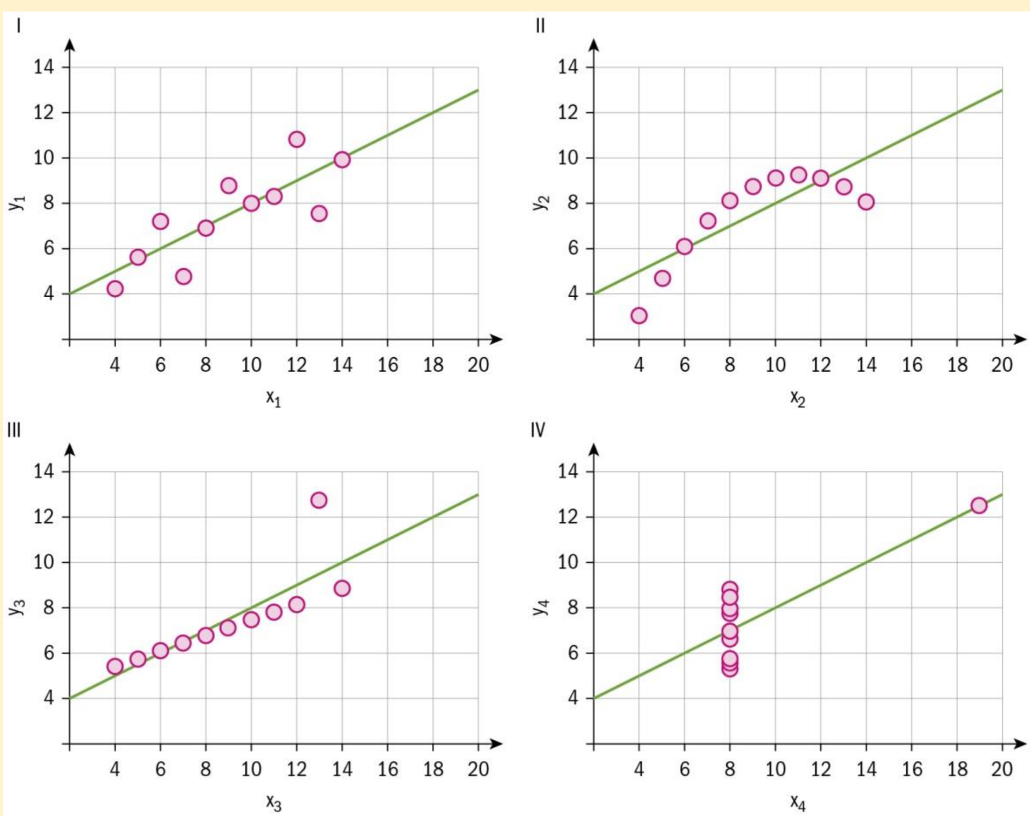
Science had to justify the cause.

Investigation 14

Conceptual understanding:

Prediction from a linear regression may be more appropriate when the data displays a clear linear trend.

1 The correlation coefficient is 0.816 for all four data sets, which is a strong correlation.



2

3 From the graphs it is clear that the data sets are very different, even though they have the same correlation coefficient. One is a 'typical' linear scatter plot, one is curved, one is almost perfectly linear with one outlier, and one appears vertical.

4 Sets I and III are appropriate to model with a linear regression, because they are linear. However, for Set III the outlier may need to be ignored. Set II is curved, and in Set IV the *y*-value does not appear to depend on the *x*-value.

5 **Conceptual:** How does the graph help us to decide whether a linear regression and Pearson's correlation coefficient are appropriate to model a data set?

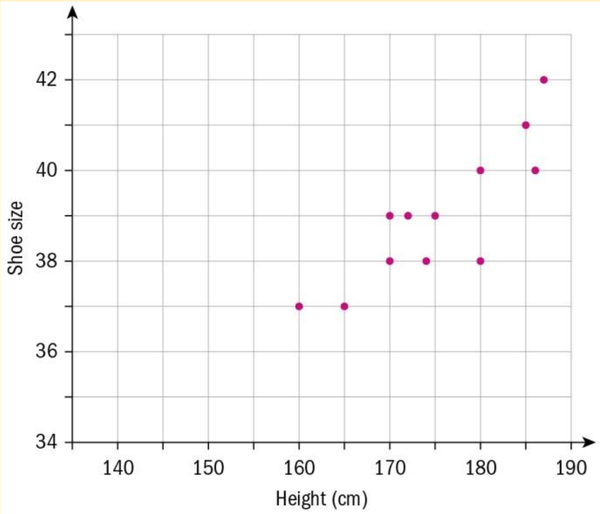
Answer: Prediction from a linear regression may be more appropriate when the data displays a clear linear trend.

Investigation 15

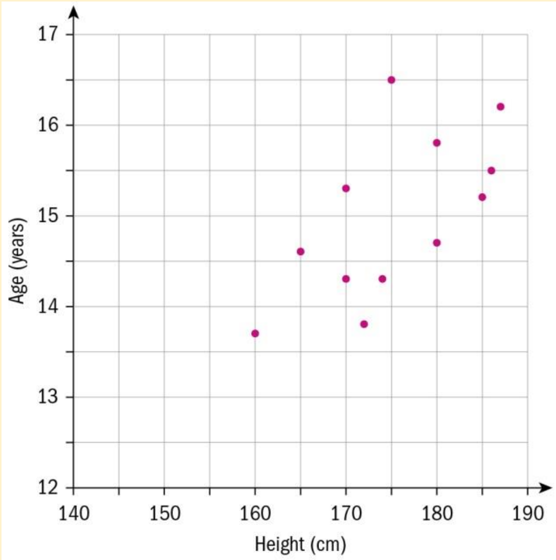
Conceptual understanding:

Prediction from a linear regression may be appropriate when the data displays a clear linear trend and this can be interpreted as a minimum of moderate strength of the Pearson's product moment correlation coefficient.

1 Shoe size as a function of height:



Age as a function of height:



Both show an approximately linear trend, so a linear regression is appropriate based on the shape of the data.

2 $r = 0.854$ for shoe size and $r = 0.637$ for age. Shoe size has a strong correlation with height whereas age has a moderate correlation.

3 **a** Shoe size: $S = 0.154h + 12.0$ (3 s.f.), age: $A = 0.0679h + 3.09$ (3 s.f.)

b **i** $S = 0.154(163) + 12.0 = 37.1$, or size 37 **ii** $A = 0.0679(163) + 3.09 = 14.2$

c Answers may vary, but students may note that there is less variation in shoe size, so it is more likely to be accurate than age; the correlation between shoe size and height is stronger; the ages in the data set near 163 had differing heights, but the same shoe size.

Students should identify shoe size as the more accurate prediction.

4 **Conceptual:** How does the strength of Pearson's correlation coefficient of a data set impact the accuracy of predictions made from its linear regression?

Answer: Predictions from linear regression are more accurate when the correlation coefficient is stronger. At least a moderate correlation and linear relationship should be established before making predictions from a linear regression.

5 **a** Solving $38 = 0.154h + 12.0$ yields $h = 169$ cm

b Regression equation of h: $h = 4.73S - 9.17$, correlation coefficient: $r = 0.854$, as before

c $h = 4.73(38) - 9.17 = 171$ cm

This prediction differs by 2 cm from using the regression of y on x.

d There is less variation in shoe size than height, so it is more difficult to predict an exact height from a known shoe size.

6 The regression of x on y is not the inverse of the regression of y on x. It is a different equation that makes different predictions.

7 **Conceptual:** Summarise: When is it appropriate to use a linear regression for prediction?

Answer: Prediction from a linear regression may be appropriate when the data displays a clear linear trend and this can be interpreted as a minimum of moderate strength of the Pearson's product moment correlation coefficient.

TOK

Correlation is the idea of modelling a pattern based on data. Causation is using data as proof that one thing causes the other.

- Does correlation need causation?
- What is a cause and effect relationship?
- Is making a model for a given situation valid as personal knowledge?

Developing inquiry skills

Has what you have learned in this chapter helped you to answer the questions from the beginning of the chapter about the taxi driver's income?

How could you apply your knowledge to investigate other business charging structures? What information would you need to find?

Thinking about the inquiry questions from the beginning of this chapter:

- Discuss if what you have learned in this chapter has helped you to think about an answer to these questions.
- Consider whether there are any that you are interested in and would like to explore further, perhaps for your internal assessment topic.

Answer: Answers will vary.

Graphs of functions – Describing the 'what' and researching the 'why'

Approaches to Learning: Thinking Skills, Communicating, Research

Exploration Criteria: Presentation (A), Mathematical communication (B) Personal engagement (C),

IB Topic: Graphs, Functions, Domain,

In this chapter students have been looking at modelling the statistical relationship between variables by considering tables of data that have then been drawn as graphs so that the relationship between the variables can be shown. They have then attempted to find the regression line (the linear algebraic function) that best fits this data. The students also looked at a few real-life situations that can be represented by an algebraic function, a table of values and also a graph of that function. As has been seen, a graph can provide a better visual representation of a data set than a function or table might. For example, on a graph it is easy to identify starting points, endpoints, rises, falls, maximums and minimums, zeros, asymptotes, etc. However, it is important that students also appreciate that relying on a graph alone has the potential for error in some cases due to inaccuracies.

The purpose of this task is to consider graphs of real-life situations and to look at how to provide a verbal description of the graphs and, perhaps more importantly, to then research why the graph may have this particular shape. This is relevant to internal assessments as students often describe the graph of data they have found without offering a critical explanation of why the graph may be as it is. More practice is then offered at the end.

Bulgaria total population

The table of data values for the graph is taken from https://www.gapminder.org/data/ (a fantastic source of world data).

To assist students when writing their paragraph ask, for example:

- What exactly does the graph show?

This is a graph of the population of Bulgaria, and how it has changed over time.

- What do the axes show and what are the units?

The axes show the year and the population in millions.

- What is the domain of the graph? What does this represent?

1800 to 2020

- Over what domain is the population rising? To what extent is it rising? When is its peak?

The population rises gently between 1800 and 1870. It then rises more rapidly and peaks at around 1989.

- Over what domain is the population falling?

The population falls from 1989 onwards.

- Add some mathematical details to these last statements. (For example, By how much did the population rise? By what percentage?)

The population rises gently between 1800 and 1870 from a value of approximately 2 million to approximately 3 million (a rise of 50%). It then rises more rapidly between 1870 and peaks at around 1989. Between these years the population rises from approximately 3 million to approximately 9 million. A 300% rise. It then begins to fall steadily from approximately 9 million to approximately 7 million. This is a fall of just over 20%.

Students should think carefully about the vocabulary that they use in their paragraph.

Here is some potentially useful vocabulary:

increased, grew, rose, went up, climbed
fell, decreased, declined, dropped
remained the same, remained constant, stabilized, levelled off
dramatically, significantly, considerably, enormously, rapidly, quickly, substantially, sharply, markedly, greatly, strongly, heavily, steeply
steadily, slightly, fractionally, gently, continuously, progressively, marginally

The interesting points and regions might be, for example, where the graph starts and ends; where the graph has a vertex (turning point); where the graph is rising (increasing) and falling (decreasing).

There is a point of inflexion at around 1870. There is a maximum at 1989.The population is rising before 1989 and falling after 1989.

Students might not know the above vocabulary yet. Encourage them to express their ideas clearly and concisely using the vocabulary that they do know. You could introduce new vocabulary if appropriate.

Possible reasons for rise: Low death rate/high birth rate, immigration

Possible reasons for decline: Emigration, high death rate/low birth rate, war/illness, natural disasters

Using sources

Students should look for articles published in scholarly journals or websites and sources that require that certain standards or criteria are met before publication.

Students should compare several opinions by scholars and experts on the topic rather than accepting the first opinion they find.

This is an ideal opportunity to address the Academic Honesty policy of the school and the referencing system that will be expected in any assignments in mathematics.

Remind students that it is important to keep a record of any sources they use. In an internal assessment, these would be needed for academic honesty, citations and bibliography.

An important skill for students to develop is the ability to evaluate the reliability of any sources they use.

Encourage students to find websites/articles etc and justify why they are reliable.

Here are some popular reasons for Bulgaria's population decline:

Most of the population moved from the villages to the cities where it is harder to raise children because the usual living space in the cities is not very large – a typical apartment is 40–80 square meters and has space for 0 to 2 children, 3 if the second is twins. People usually own their apartments so it's hard to move when the child arrives. This means that the quality of life decreases with each new family member.

All forms of child care, kindergartens, social security are not well developed and don't help parents.

Pre-1989 the borders were closed and people weren't allowed to leave.

The economy forced people to leave to seek employment and a better life.

Young people pursued further education abroad. Few of them came back.

The age at which people have their first child is increasing due to cultural changes. Many women prefer to finish their education and build a career first.

For an extension, students could also try to find a function that best fits this data in order to make predictions of future values. When students predict what will happen to the population of Bulgaria in the future, they should be aware of the dangers of extrapolation.

Worldwide Wii Console Sales

Again, students could use the questions in the box at the start of this spread as guidance if needed.

Encourage students to Include some mathematical detail in their descriptions.

The domain is from 2006 to 2016.

The graph is increasing between 2006 and 2008.

The graph is then decreasing between 2008 and 2016.

Students should initially think of some possible reasons for the changes and trends and then research the actual reasons.

You could ask students:

- Why do you think this is the domain of the data?
- What are some interesting years and periods of time that it might be interesting to investigate?

The Wii was launched in September 2006 and was one of the biggest selling game consoles for a short time before declining from 2008. In 2016 Nintendo stopped producing and selling the Wii.

This type of sales shape is not unusual for console sales (and other technology) – there are similar patterns of peaking in the 2nd/3rd year and then declining for PS2 and Gamecube, for example. Console hardware becomes outdated and potential buyers already own the device.

The peak coincides with the iPhone release and 'the death of TV-based devices', perhaps.

Global mean temperature anomaly

Note: Moving averages and linear trend lines are not in the current syllabus.

However, this is still a good topic for practicing research and could also be the type of presentation technique that they could encounter or use in their Internal Assessment. This type of presentation is suitable for demonstrating and helping to illustrate and explain trends and the mathematics involved is at a level commensurate with the SL course.

A moving average is a technique used to get an overview of the trends in a data set. It is an average of any subset of numbers.

A linear trend line is a straight line of best fit. It generally shows that something is increasing or decreasing at a steady rate.

Students could consider CO_2 emissions, volcanic eruptions, El nino events etc.

Here is an interesting article giving a possible reason for the drop in the 1940s:

https://www.newscientist.com/article/dn14006-buckets-to-blame-for-wartime-temperature-blip

Again, encourage students to think about the reliability of any sources they use.

You could ask:

- Is this a reliable source?

Students may wish to compare this data with other data that can be found on the internet and try to explain any possible differences. (Different collection methods, slightly different data being collected, biases, etc).

Extension

This extension encourages students to do a number of things.

First, they need to look for an appropriate graph. Appropriate here means interesting and relevant.

They also have to describe and explain the shape of the graph for themselves by researching.

Finally, by writing a series of questions they are thinking about the communication of the ideas behind the reason. By thinking about what questions they need to ask, they are also thinking about what areas they need to explain for themselves.

Examples of question students could ask are:

- What explanations can you give for the sudden rise/fall in the data between … and …?

5 Quantifying uncertainty: probability

Essential understandings

Probability enables us to quantify the likelihood of events occurring and so evaluate risk. Both statistics and probability provide important representations which enable us to make predictions, valid comparisons and informed decisions. These fields have power and limitations and should be applied with care and critically questioned to differentiate between the theoretical and the empirical/observed. Probability theory allows us to make informed choices, to evaluate risk, and to make predictions about seemingly random events.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Approximation in data can approach the truth but may not always achieve it.
- Correlation and regression are powerful tools for identifying patterns and equivalence of systems.
- Modelling and finding structure in seemingly random events facilitates prediction.
- Different probability distributions provide a representation of the relationship between the theory and reality, allowing us to make predictions about what might happen.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
Standard Level	
Random behavior by nature involves unpredictability in the short term.	Investigation 1
Random behavior by nature involves unpredictability in the short term. There is a long-term trend towards the theoretical probability, but the data from an experiment can diverge from the theoretical at any point. The number given by the formula for theoretical probability serves as a good model for quantifying the likelihood of the event and allows us to make predictions.	Investigation 2
Representing combined probabilities using diagrams can show structure and pattern that a formula may not.	Investigation 3
Mutually exclusive events cannot occur at the same time, which results in no intersection. Independent events represent unrelated separate events in which the probability of one event does not affect the outcome of the other. However, mutually exclusive events cannot be independent.	Investigation 4
When calculating the probability of an event, it may be more efficient to look at the probability of the complementary event.	Investigation 5

Syllabus sections covered in this chapter:

- SL4.5*
- SL4.6*
- AHL4.19

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 204: Quantifying uncertainty: probability	Page 219: Example 7 Page 219: Example 8		Pages 210, 215, 220, 223

Assessment opportunities

End of chapter summary	Chapter review	Exam-style questions
Page 224	Page 226	Page 227

Opening problem

Two taxi companies operate in Mathcity: Blackcabs and Yellowrides. 85% of the cabs in the city work for Blackcabs and are coloured black. The rest of the cabs in the city work for Yellowrides and are coloured yellow. A taxi was involved in a hit and run accident at night. A witness made a statement to the police that the taxi involved in the accident was yellow. The court carried out a series of tests of the reliability of the witness, asking her to identify the colour of a random sequence of taxis. The witness correctly identified each one of the two colours 80% of the time and failed 20% of the time. What is the probability that the taxi involved in the accident was yellow?

Answer: $\frac{12}{29} \approx 0.414$

Developing inquiry skills

Write down any similar inquiry questions you might ask if you were asked to predict the reliability of the witness if 50% of the cars in the city were Yellowrides or if another taxi company Blue Taxis also operated in Mathcity. What questions might you need to ask in these scenarios?

Answers: How could the diagram for the original problem be adapted? Would the witness pay closer attention to Yellow cabs when they were less likely (than 50%) to appear? Would the witness find black and blue harder to distinguish than black and yellow?

The opening problem is frequently answered incorrectly and is an example of the **base rate fallacy** which you may like to explore.

This problem has a different context but the same structure:

Designers of medical tests have to quantify "false positives", in which a person tests positive for a disease but in fact does not have the disease, and "false negatives", in which a person tests negative for a disease but in fact does have the disease.

Clinical trials reveal that in a large population, a disease affects 0.4% of the population. A medical test has been found to be accurate in that the frequencies of false positives and false negatives are small. The probability that the test says you have the disease when you do not in fact have the disease is 0.03. The probability that the test says you have not got the disease but in fact you do is 0.02.

You take the test and are told that you have the disease. What is the probability that you actually do have the disease?

5.1 Reflecting on experiences in the world of chance. First steps in the quantification of probabilities

TOK

Intuition is often not a good way of knowing in Probability. There are lots of other examples in this field where intuition potentially lets you down. What about in other areas of knowledge? Is it more or less reliable there?

Investigation 1

Conceptual understanding:

Random behavior by nature involves unpredictability in the short term.

Event D is deliberately set up to be easy to agree on. However, others are far more challenging to agree on. Students should conclude that assigning probability values derived from personal knowledge, interests and beliefs is subjective. Students may reflect on Plato's definition of knowledge as "Justified true belief". A particular probability may be justified in part by one's feelings, intuitions and experiences.

Investigation 2

Conceptual understandings:

Random behaviour by nature involves unpredictability in the short term.

There is a long-term trend towards the theoretical probability, but the data from an experiment can diverge from the theoretical at any point.

The number given by the formula for theoretical probability serves as a good model for quantifying the likelihood of the event and allows us to make predictions.

2 Instructions for the spreadsheet:

EXPERIMENT: throwing a 12-sided die and recording if the uppermost number is prime (EXCEL)

Type 1 into cell A1, 2 into cell A2, 3 into cell A3. Select cells A1, A2 and A3 and drag down to A100 to get all the numbers 1..100 in column A. These numbers are the number of trails.

Type "=RANDBETWEEN(1,12)" in cell B1 and drag down to cell b100. These generate random numbers between 1 and 12, and so they simulate throwing a 12-sided die. Press F9 to throw the 100 dice again.

Type "=OR(B1=2,B1=3,B1=5,B1=7,B1=11) into cell C1. This gives "TRUE" if the number B1 is prime, FALSE otherwise. Drag down to cell C100.

Type "=IF(C1,1,0)" in cell D1. This just changes "TRUE" to 1 and "FALSE" to zero. Drag down to cell D100.

Type "=D1" into cell E1

Type "=E1+D2" into cell E2 and drag down to cell E100. This gives the cumulative number of times that a prime number was thrown.

Type "=E1/A1" into cell F1 and drag down to cell F100. This gives the experimental probability of throwing a prime number when the experiment is repeated in 100 trials

Add a graph showing A1..A100 on the *x*-axis and F1..F100 on the *y*-axis.

3 and **4**Answers will vary.

5 **Factual:** What is the set of all possible values of theoretical probabilities?

Answer: Theoretical probabilities always lie in the domain [0,1].

6 The graph should fluctuate above and below the line randomly.

7 **Factual:** What is the relationship between relative frequency and theoretical probability in the short term?

Answer: In the short term, we can't know when exactly relative frequency (experimental probability) may equal the theoretical or for how long it is more or less than the theoretical.

8 **Factual:** What is the relationship between relative frequency and theoretical probability in the long term?

Answer: In the long term, relative frequency (experimental probability) approaches the theoretical. However, it can diverge from the theoretical probability at any time.

9 **Conceptual:** Does random behaviour involve predictability in the short term or unpredictability?

Answer: Random behaviour by nature involves unpredictability in the short term.

10 **Conceptual:** Does random behaviour involve predictability in the long term or unpredictability?

Answer: Both. There is a long-term trend towards the theoretical probability, but the data from an experiment can diverge from the theoretical at any point.

11 **Conceptual:** How may we interpret and apply the number quantified by the formula for the theoretical probability of an event?

Answer: The number given by the formula for theoretical probability serves as a good model for quantifying the likelihood of the event and allows us to make predictions.

TOK

The game in the St Petersburg paradox follows these rules. You flip a coin; if it lands tails up then you lose and the game is over. If the coin lands heads up then you win one rouble and the game continues. The coin is tossed again. If it is tails, then the game ends and you keep the money you have won. If it is heads, then you win an additional two roubles.

For each successive head you double your winnings from the previous round, but, at the first tail, the game is over.
How much would you pay to play this game?

Developing inquiry skills

There are four outcomes in the first opening scenario:

- a taxi is yellow and is identified as yellow
- a taxi is yellow and is identified as black
- a taxi is black and is identified as yellow
- a taxi is black and is identified as black.

Are these equally likely outcomes?

In 1000 trials, how many occurrences of each outcome would you expect?

Answer: No they are not equally likely outcomes. 170, 680, 120 and 30 respectively.

5.2 Representing combined probabilities with diagrams

Investigation 3

Conceptual understanding:

Representing combined probabilities using diagrams can show structure and pattern that a formula may not.

2 P(Bio and Chem)= $\frac{3}{8}$

3 Answers will vary.

5 P(total is prime)= $\frac{3}{8}$

6

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12
8	9	10	11	12	13	14

Students can see from the pattern that for example the outcomes 11 and 5 are equiprobable.

7 **Conceptual:** What advantages are there in using a diagram in problem solving with combined probabilities?

Answer: Representing combined probabilities using diagrams can show structure and pattern that a formula may not.

TOK

A good question for a debate or blog post would be: "Ethics is an area of knowledge in its own right. Where do you see an intersection of the areas of knowledge of mathematics and ethics?" When taking a chance decision in your life, on which skills do you rely and in which order? Intuition (your gut feeling)? Reason? Emotion? Memory? Faith? Imagination?

Developing inquiry skills

In the first opening scenario, imagine 100 trials. How many outcomes would you expect in each area shown on this diagram?

Answer:

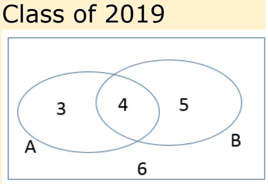
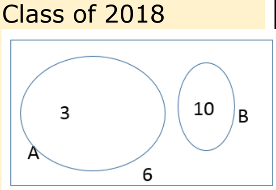
		Cab yellow?	
		Yes	**No**
Witness correct?	**Yes**	12	68
	No	3	17

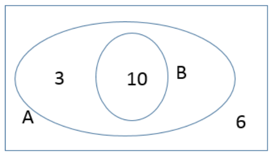
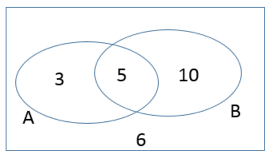
5.3 Representing combined probabilities with diagrams and formulae

Investigation 4

Conceptual understanding:

Mutually exclusive events cannot occur at the same time, which results in no intersection. Independent events represent unrelated, separate events in which the probability of one event does not affect the outcome of the other. However, mutually exclusive events cannot be independent.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Venn diagram	P(A)	P(A′)	P(B)	$P(A\cap B)$	$P(A\cup B)$	$P(A\mid B)$	P(A) + P(B)	P(A) x P(B)	P(A) +P(B) − $P(A\cap B)$	$\frac{P(A\cap B)}{P(B)}$
Class of 2019 	$\frac{7}{18}$	$\frac{11}{18}$	$\frac{1}{2}$	$\frac{2}{9}$	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{8}{9}$	$\frac{7}{36}$	$\frac{2}{3}$	$\frac{4}{9}$
Class of 2018 	$\frac{3}{19}$	$\frac{16}{19}$	$\frac{10}{19}$	0	$\frac{13}{19}$	0	$\frac{13}{19}$	$\frac{30}{361}$	$\frac{13}{19}$	0

Class of 2017 	$\frac{13}{19}$	$\frac{6}{19}$	$\frac{10}{19}$	$\frac{10}{19}$	$\frac{13}{19}$	1	$\frac{23}{19}$	$\frac{130}{361}$	$\frac{13}{19}$	1
Class of 2016 	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{5}{8}$	$\frac{5}{24}$	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{23}{24}$	$\frac{5}{24}$	$\frac{3}{4}$	$\frac{1}{3}$

1 Each pair of probabilities adds to 1.

2 Because the union of two complementary events is the whole sample space.

3 The probabilities in each pair are equal.

4 The intersection must be subtracted or else it will be counted twice.

5 The probabilities in each pair are equal.

6 Factual: Which Venn diagram shows "mutually exclusive events"?

Answer: The class of 2018

7 Factual: Which Venn diagram shows "independent events"?

Answer: The class of 2016

8 Conceptual: What is the difference between mutually exclusive and independent events? Can mutually exclusive events be independent? Why or why not?

Answer:

Mutually exclusive events cannot occur at the same time, which results in no intersection. Independent events represent unrelated, separate events in which the probability of one event does not affect the outcome of the other. However, mutually exclusive events cannot be independent.

Reflect:

Are complementary events independent events? **Answer:** No

Are complementary events mutually exclusive? **Answer:** Yes

Can mutually exclusive events be independent? **Answer:** No

Can non-mutually exclusive events be independent? **Answer:** Yes

Developing inquiry skills

Which of the events in the first opening scenario are independent? Which are mutually exclusive?

Answer: The witness identifying the colour of the cab correctly is independent of the colour of the cab. The cab is either black or yellow so these are mutually exclusive events, just as the witness being right or wrong are mutually exclusive events.

5.4 Complete, concise and consistent representations

TOK

The difference between personal knowledge and shared knowledge offers you the chance to consider the difference between "what I know" and "what we know".
Shared knowledge may come from texts, teachers, media etc, personal knowledge is gained through the experience of the individual, such as ice is cold or rabbits are fluffy.
Somebody who studies computing might view their laptop differently because of their academic knowledge. Their personal knowledge had been affected by the shared knowledge they had gained in class. This would be an intersection of the two types of knowledge.

Investigation 5

Conceptual understanding:

When calculating the probability of an event, it may be more efficient to look at the probability of the complementary event.

1 6

2 3

3 **Factual:** Hence fill in the table:

Answer:

Number of dice	**How many probabilities would be found and added from the tree diagram representation**	**How many probabilities would be found, added and their total subtracted from 1 using a complementary events representation.**
2	2	2
3	6	3
4	12	4
5	20	5
k	$k^2 - k$	k

4 **Factual:** Can you generalise for n_1, n_2 ….. n_k dice with distinct colours c_1, c_2 … c_k ?

Answer: It's more efficient to use the complementary events representation since that would involve working out k probabilities instead of the $k^2 - k$ probabilities that would be necessary for the tree diagram representation.

5 **Conceptual:** Why is it useful to calculate probabilities for the complementary event in some situations?

Answer: When calculating the probability of an event, it may be more efficient to look at the probability of the complementary event.

TOK

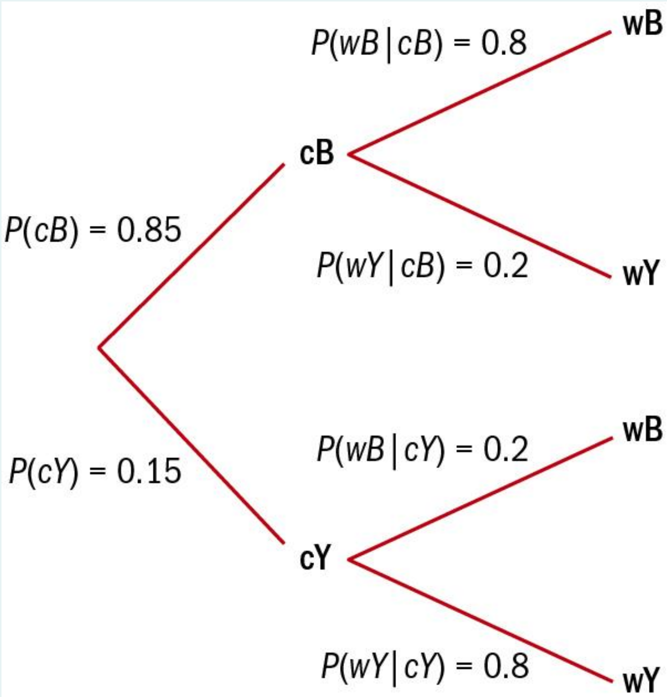
Which option do you think is more likely? Why? We will return to this question at the end of this section.
Consider the ethics of gambling.

Developing inquiry skills

Apply what you have learned in this section to represent the first opening problem with a tree diagram. Hence find the probability that a cab is identified as yellow. Apply the formula for conditional probability to find the probability that the cab was yellow given that it was identified as yellow. How does your answer compare to your original subjective judgement?

Answer: cB is the event that the cab was in fact black, cY is the event that the cab was in fact yellow.

wB is the event that the cab was identified as black by the witness, wY is the event that the cab was identified as yellow by the witness.



Hence $P(wY) = 0.85 \times 0.2 + 0.15 \times 0.8 = 0.29$

Hence $P(cY | wY) = \dfrac{P(cY \cap wY)}{P(wY)} = \dfrac{0.8 \times 0.15}{0.29} \approx 0.414$

Random walking!

Approaches to Learning/learner profile: Critical thinking

Exploration Criteria: Mathematical communication (B), Personal engagement (C), Use of mathematics (E)

IB Topic: Probability, Discrete Distributions

This problem is designed to encourage students to think of simulation as a reasonable and acceptable approach to probability problems that may be too difficult to approach theoretically as they develop. This problem has a clearly stated aim and is accessible at first using Mathematical communication (Criterion B) and methods familiar to students from the chapter. The mathematics required to prove the result is quite difficult for some, but this should not restrict students from accessing it and using the tools available to them.

At the beginning of the problem students use basic coin tossing simulations and collect results as a class to produce more results and a hopefully more accurate answer. At the end of the task students are asked to consider using computer simulation. Coding for this is not only accessible for a computer science student or experienced programmer but can be learnt with a little effort and

Personal engagement (Criterion C). Further personal engagement can be shown by extending the problem once the code has been mastered.

The proof of the result requires some understanding of probability distributions so this task may be better covered after that.

The problem

The problem is adapted from a famous problem in a branch of mathematical problems involving 'random walks'. Study of this branch has contributed to many different areas in physics and chemistry (Brownian motion and diffusion), biology (genetics, animal movements, population dynamics), economics (modeling share prices) and computer science (social media suggestions), amongst others.

Explore the problem

Since the man moves left or right with equal probability, a coin toss can be used to simulate this.

If appropriate, ask:

- Why is a coin toss a suitable simulation?

Students play the game 10 times and find the average number of steps taken. Discuss why this may not be an accurate result.

Ask:

- What could you do to improve the accuracy of the average?

Discuss the improved result based on a larger sample size.

The average may be getting closer, but you do not know if this is the actual number.

You can only be certain by proving the result theoretically.

Ask:

- What have you noticed so far?

(For example, always an odd number of steps, theoretically could go on forever, etc.)

Calculate probabilities

Students may need help when constructing the tree diagram.

Ask:

- What are the limitations of using a tree diagram in this case?

The tree diagram is very large! It becomes impossible to draw after 6 or 7 tosses.

Remind students how to use a tree diagram to find probabilities.

If needed, to help students find the probability that the man falls into the ditch after a total of exactly 5 steps, ask:

If the man moves left (L) then left again (L) and then right (R) and then left (L) and then left (L) then he will be in the ditch. Where is this scenario on your tree diagram?

What is the probability that the man takes this particular sequence of steps? In other words what is the probability of TTHTT?

Probability is (0.5)5 = 0.03125 = (1/32)

What other sequences of coin tosses will lead to the man falling into the ditch after exactly 5 steps?

TTHTT, THTTT, THHHH, HTTTT, HTHHH, HHTHH

What are the probabilities associated with each of these sequences?

They are all (0.5)5 = 0.03125 = 1/32

What is the probability that the man falls into the ditch after a total of exactly 5 steps?

6(0.5)5 = 6(0.03125) = 3/16

Minimum number of steps to fall into the ditch is 3.

Maximum is infinite.

Probability that the man falls into the ditch after a total of exactly 3 steps is

$$2(0.5)3 = 2(0.125) = \frac{1}{4}$$

To explain why all the paths have an odd number of steps:

It is an odd number of steps because:

From the centre, after the first step, the man will always be one step away from the centre (two steps away from the ditch on that side).

From here, after 2 steps, he will either be in the ditch on the same side, back to the same position, or one step from the centre (two steps away from the ditch) on the other side. This will repeat.

This gives 1 + a multiple of 2 which is odd.

You could use a diagram to demonstrate this, with the centre shown by a black dot, one step from the centre on either side shown by a red dot, two steps away from the centre on each side shown by a black dot, and the ditches shown by blue dots.

The probabilities are:

x	1	2	3	4	5	6	7	8	9	10	11	12	....
P(X = x)	0	0	$\frac{1}{4}$	0	$\frac{3}{16}$	0	$\frac{9}{64}$	0	$\frac{27}{256}$	0	$\frac{81}{1024}$	0	

The table alternates between 0 and a value.

The next value is calculated by multiplying by $\frac{3}{4}$.

Ask:

- Can you explain where the value of $\frac{3}{4}$ comes from in this situation?

Again, you could use diagrams to demonstrate this.

Simulation

Students will study expectation in a chapter **13**.

For now, if appropriate, you could ask:

- What is an expected value?

You could also give students the formula for calculating the expected number of steps that would be required:

$$E(X) = \sum_{1}^{\infty} x\mathrm{P}(X = x)$$

E(X) is the expected number of steps.

This will give you the exact theoretical answer to the problem posed.

Help them to understand and use this formula if needed.

The formula is:

Multiply each probability by its related value and sum the result.

Students may note here that this problem will be complicated because there are an infinite number of values of *x* that will result in falling into the ditch.

It is possible to calculate the expected value, but it requires mathematics that will be beyond the SL and HL syllabus.

As extension, students could perhaps try to do this when they have completed Chapter **13** and fancy a challenge!

Here is the solution written out:

$$E(X) = \sum_{x=1}^{\infty} x * P(X = x)$$

$$= (1*0) + (2*0) + (3*\frac{1}{4}) + (4*0) + (5*\frac{3}{16}) + (6*0) + (7*\frac{9}{64}) + \ldots$$

$$= (3*\frac{1}{4}) + (5*\frac{3}{16}) + (7*\frac{9}{64}) + (9*\frac{27}{264}) + \ldots$$

$$= \frac{1}{4}(3 + 5*\frac{3}{4} + 7*\frac{9}{64} + \ldots)$$

$$= \frac{1}{4}(3 + 5*\frac{3}{4} + 7*(\frac{3}{4})^2 + 9*(\frac{3}{4})^3 + \ldots)$$

This is an infinite Arithmetico-Geometric series. Its sum can be found neatly as follows:

$$E(X) = \frac{1}{4}[3 + 5*(\frac{3}{4}) + 7*(\frac{3}{4})^2 + 9*(\frac{3}{4})^3 + \ldots]$$

$$\frac{3}{4}*E(X) = \frac{1}{4}*[3*(\frac{3}{4}) + 5*(\frac{3}{4})^2 + 7*(\frac{3}{4})^3 + 9*(\frac{3}{4})^4 + \ldots]$$

Subtracting,

$$\frac{1}{4}E(X) = \frac{1}{4}[3 + 2*(\frac{3}{4}) + 2*(\frac{3}{4})^2 + 2*(\frac{3}{4})^3 + \ldots]$$

$$E(X) = 3 + 2*(\frac{3}{4}) + 2*(\frac{3}{4})^2 + 2*(\frac{3}{4})^3 + \ldots$$

$$E(X) = 3 + 2*\frac{3}{4}*[1 + (\frac{3}{4}) + (\frac{3}{4})^2 + (\frac{3}{4})^3 + \ldots]$$

$$= 3 + 2*\frac{3}{4}*\frac{1}{1-\frac{3}{4}}$$

$$= 3 + 2*\frac{3}{4}*4$$

$$= 9$$

This shows that the theoretical result is 9. Compare this with the result from the earlier simulation by the class.

At this stage, you could tell the class that the expected number of steps is 9.

Ask:

- Why do you think simulations are used?

Collecting and recording large numbers of results by hand is very time consuming and can be very expensive.

Here are some examples of computer coding that could be copied or shown to the students. It would be possible to replicate these results on most computer coding systems. (If there are computer science students in the class then they could be encouraged to help the class, although simple coding should be accessible to all!)

Note: The idea here is that simple computer coding is accessible to all students with a little work, and then more complicated problems can be solved. This is considerably more efficient than more manual methods. It is also worth pointing out that this is actually an exceptionally real process in many, many fields of work such as meteorology, disaster management, economics and finance, sports predictions, etc. This is one of the real avenues of work for the modern mathematician!

Extension

Suggestions of how students could vary the problem:

- Change the start position.
- Use a bias coin.
- Change the number of steps from the centre to the ditch.
- Change the problem to 2 dimensions.

Students may also be able to devise their own probability question which they could answer using simulation.

6 Modelling relationships with functions: power and polynomial functions

Essential understandings

Models are depictions of real-life events using expressions, equations or graphs while a function is defined as a relation or expression involving one or more variables. Creating different representations of functions to model the relationships between variables, visually and symbolically as graphs, equations and tables represents different ways to communicate mathematical ideas.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- The parameters in a function or equation correspond to geometrical features of a graph and can represent physical quantities in spatial dimensions.
- Moving between different forms to represent functions allows for deeper understanding and provides different approaches to problem solving.
- Our spatial frame of reference affects the visible part of a function and by changing this "window" can show more or less of the function to best suit our needs.
- Different representations of functions, symbolically and visually as graphs, equations and tables provide different ways to communicate mathematical relationships.
- The parameters in a function or equation correspond to geometrical features of a graph and can represent physical quantities in spatial dimensions.
- Moving between different forms to represent functions allows for deeper understanding and provides different approaches to problem solving.
- Our spatial frame of reference affects the visible part of a function and by changing this "window" can show more or less of the function to best suit our needs.
- Changing the parameters of a trigonometric function changes the position, orientation and shape of the corresponding graph.
- Different representations facilitate modelling and interpretation of physical, social, economic and mathematical phenomena, which support solving real-life problems.
- Extending results from a specific case to a general form and making connections between related functions allows us to better understand physical phenomena.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	Investigation
In order for an inverse function to exist or be invertible a function must map only one independent variable with one dependent variable and this can be the case within a largest possible domain or over a restricted domain.	Investigation 1
The parameters of the quadratic function alter the symmetry, vertex and intercepts and distinguish geometrical features of a parabola such as concavity.	Investigation 2
Cubic functions can be used to model real-life data, while displaying both a minimum and a maximum value.	Investigation 3
The graph of positive power functions goes through the origin and displays features such as symmetry while the graph of negative power functions displays features such as symmetry and asymptotic behavior.	Investigation 4

The inverse proportional function can be used to model real-life data with asymptotic behavior in which, as the dependent variable increases, the independent variable decreases.	Investigation 5 Investigation 7
The inverse proportional function displays features such as symmetry and asymptotes, a vertical and a horizontal line towards which the graph approaches infinitesimally closely, without ever reaching it.	Investigation 6
Asymptotes occur when a value leads to division by zero.	Investigation 8

Syllabus sections covered in this chapter:

- SL2.2*
- SL2.3*
- SL2.4
- SL2.5
- SL1.8
- SL2.6
- AHL2.7
- AHL2.8
- AHL2.9

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 232: Modelling relationships with functions: power and polynomial functions	Page 243: Example 4 Page 245: Example 5 Page 248: Example 6 Page 256: Example 7	Page 235: Example 1 Page 238: Example 3 Page 245: Example 5 Page 248: Example 6 Page 259: Example 9 Page 264: Example 10 Page 269: Example 11	Pages 244, 257, 265, 277

Assessment opportunities

End of chapter summary	Chapter review	Exam-style questions
Page 279	Page 281	Page 284

6.1 Quadratic models

TOK

A 500-word response on paper or in a blog is a suitable way of answering this TOK question.

Answers might include that mathematics allows humans to pass on knowledge from one another, across national, cultural and religious borders. The social communication required for the symbology and language to develop is a form of communication where people interact with one another whilst passing on the knowledge of mathematics.

A counterclaim might be that many mathematicians (e.g. Andrew Wiles) work best in isolation.

Investigation 1

Conceptual understanding:

In order for an inverse function to exist or be invertible a function must map only one independent variable with one dependent variable and this can be the case within a largest possible domain or over a restricted domain.

1 $0 \le t \le 6$, so that the function is one-to-one.

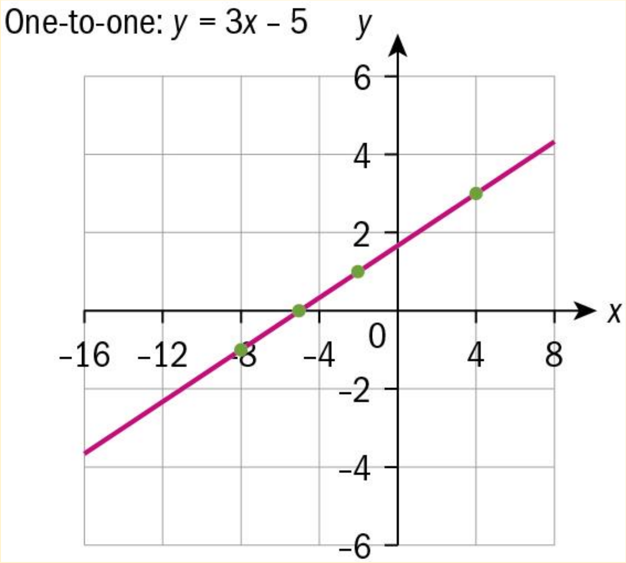
2 $h = 180 - 5 \cdot (t-6)^2 \Rightarrow 5 \cdot (t-6)^2 = 180 - h \Rightarrow (t-6)^2 = \dfrac{180-h}{5} \Rightarrow t - 6 = \pm\sqrt{\dfrac{180-h}{5}}$

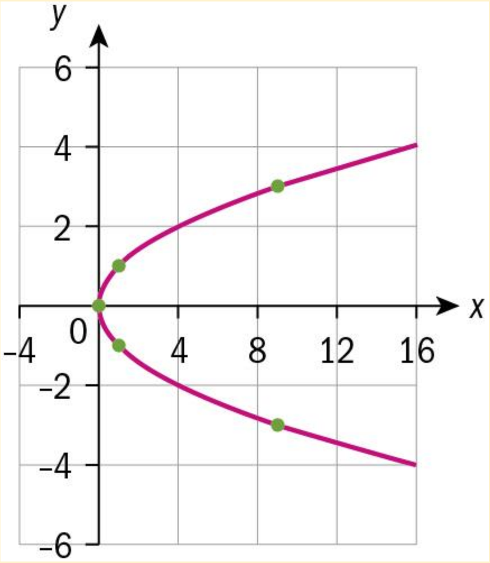
$\Rightarrow \quad t = 6 \pm \sqrt{\dfrac{180-h}{5}} \Rightarrow t = 6 - \sqrt{\dfrac{180-h}{5}}$ since $0 \le t \le 6$.

3 The domain becomes the range and vice versa. The restricted domain from part **1** helps in deciding which sign to choose. In this case the negative sign was chosen.

4 **a** In order to sketch the graph of the inverse relation, use (x_o, y_o) points from the original graph and plot them at (y_o, x_o).

One-to-one: $y = 3x - 5$ Many-to-one: $y = x^2$





b The one-to-one's inverse relation is a function since each x-value from the domain corresponds to a single y-value. On the other hand, the many-to-one's inverse relation is not a function, since there exists at least one x-value in the domain which corresponds to more than one y-value. For example when $x = 1$, there are two points on the graph: points $(1,1)$ and $(1,-1)$.

5 **Conceptual:** Why is a function not always invertible (invertible means that its inverse function exists), and when is it possible for a function to be invertible?

Answer: In order for an inverse function to exist or be invertible a function must map only one independent variable with one dependent variable and this can be the case within a largest possible domain or over a restricted domain.

TOK

Questions that might be used include:

- Should mathematics be used to take lives?
- What are the decisions to be made when developing the accuracy and stealth of weapons such as drones?

Consider the role of the scientists developing the atomic bomb.

6.2 Problems involving quadratics

Investigation 2

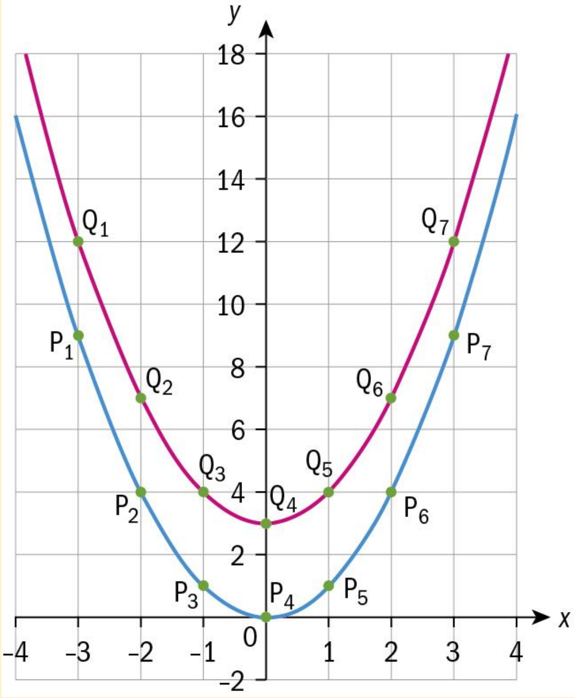
Conceptual understanding:

The parameters of the quadratic function alter the symmetry, vertex and intercepts and distinguish geometrical features of a parabola such as concavity.

1 a

x	−3	−2	−1	0	1	2	3
$f(x)$	9	4	1	0	1	4	9
$g(x)$	12	7	4	3	4	7	12

b



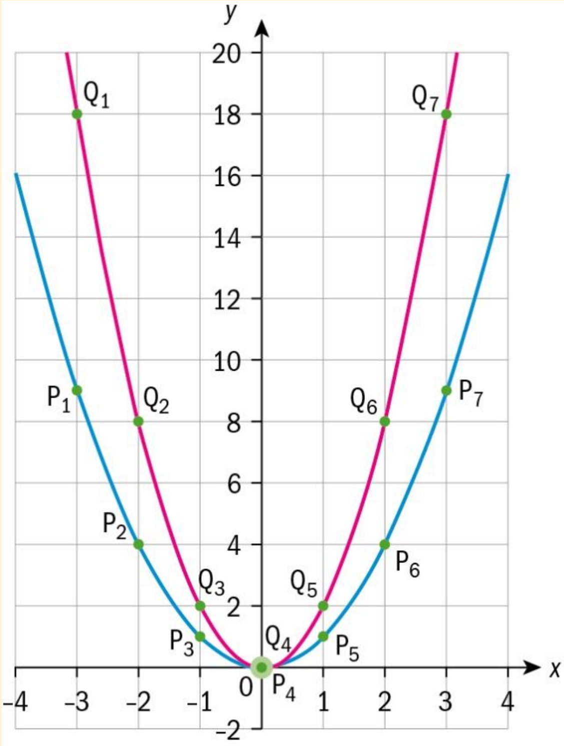
c Translates the graph 3 units upwards. The *x*-coordinates remain the same. The *y*-coordinates of all the points become 3 units more from what they were before.

d It would translate it downwards.

2 a

x	−3	−2	−1	0	1	2	3
$f(x)$	9	4	1	0	1	4	9
$g(x)$	18	8	2	0	2	8	18

b



c Stretches the graph vertically away from the x-axis. The x-coordinates remain the same. The y-coordinates of all the points become double of what they were before.

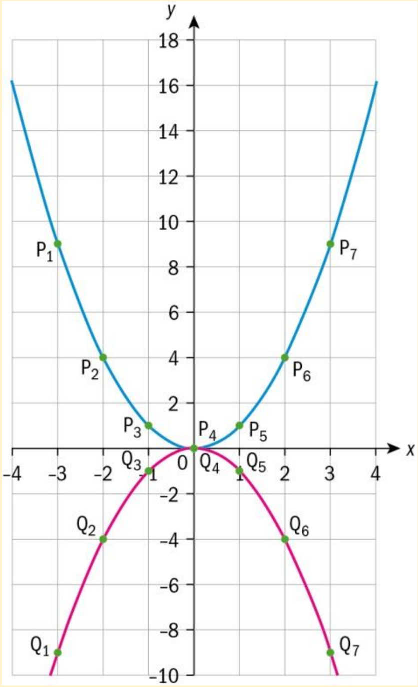
d Points on the x-axis.

e It would "shrink" the graph vertically towards the x-axis.

3 a

x	−3	−2	−1	0	1	2	3
$f(x)$	9	4	1	0	1	4	9
$g(x)$	−9	−4	−1	0	−1	−4	−9

b



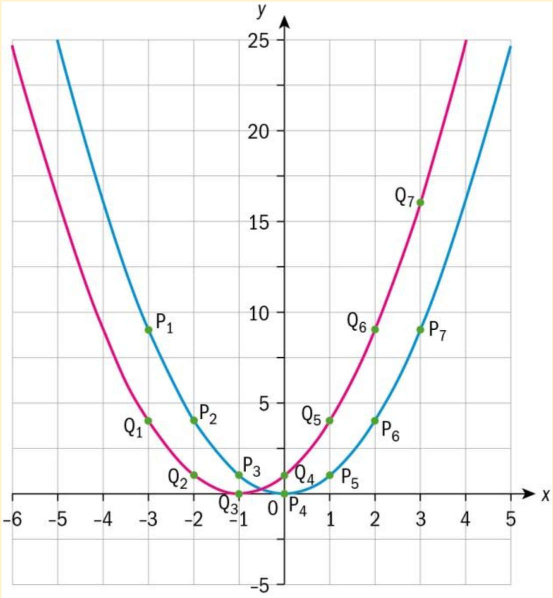
c Reflects the graph about the *x*-axis. The *x*-coordinates remain the same. The *y*-coordinates of all the points become opposite of what they were before.

d Points on the *x*-axis.

4 a

x	−4	−3	−2	−1	0	1	2
$f(x)$		9	4	1	0	1	4
$g(x)$	9	4	1	0	1	4	9

b



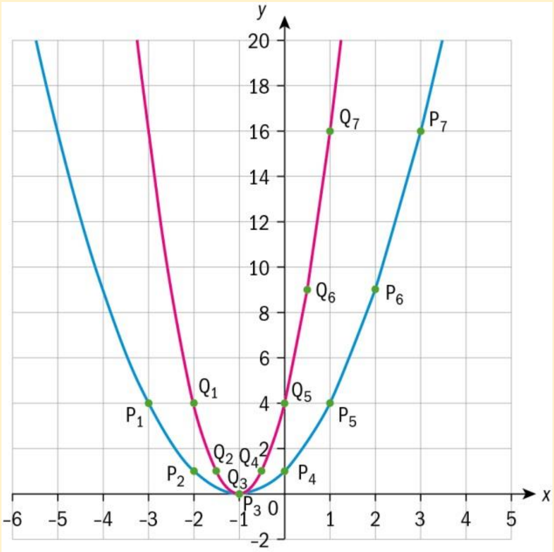
c Translates the graph 1 unit to the left. The x-coordinates of all the points become 1 unit less than what they were before. The y-coordinates remain the same.

d It would translate the graph to the right.

5 a

x	−4	−3	−2	−1.5	−1	−0.5	0	0.5	1	2
$f(x)$	9	4	1		0		1		4	9
$g(x)$			9		1	0	1	4	9	

b



c Shrinks the graph horizontally towards the y-axis. The x-coordinates of all the points become half of what they were before. The y-coordinates remain the same.

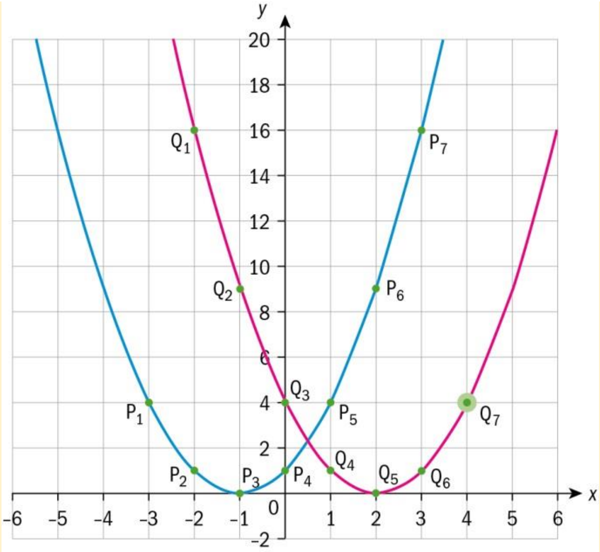
d The ones on the y-axis.

e It would stretch the graph away from the y-axis.

6 a

x	−4	−3	−2	−1	0	1	2	3	4
$f(x)$	9	4	1	0	1	4	9		
$g(x)$			9	4	1	0	1	4	9

b



c Reflects the graph about the *y*-axis. The *x*-coordinates of all the points become opposite of what they were before. The *y*-coordinates remain the same.

d Those on the *y*-axis.

7 **Conceptual:** What is the effect of changing the parameters of the quadratic function $y = a(bx + c)2 + d$ on its graph?

Answer: The parameters of the quadratic function alter the symmetry, vertex and intercepts and distinguish geometrical features of a parabola such as concavity.

TOK

A chance to explore the limitations of graphs and charts in delivering information about functions and phenomena in general, relevance of modes of representation.

A good question to ask for a response to is:

- Should we accept simplicity over accuracy to relay information?

6.3 Cubic functions and models

TOK

Questions to start a discussion in class might include:

- Is there a difference in accuracy between geometrical and algebraic representation?
- Which forms of representation stay in your memory for longer? Formulas, diagrams, colours? Why?

Investigation 3

Conceptual understanding:

Cubic functions can be used to model real-life data, while displaying both a minimum and a maximum value.

1 The width of the card is 10 cm and the length is 12 cm. When you cut a square of length x cm from each corner, you are left with a rectangle in the centre of the card which measures (10 – 2x) cm by (12 – 2x) cm. This rectangle forms the base of the box.

You then fold up the four rectangles on the sides of the base to make the sides of the box. Each of these has height x cm, so the box has height x cm.

2 $V = x(12-2x)(10-2x)$

4 The function cannot have an inverse since it is not a one-to-one function under the maximum possible domain which is $0 < x < 5$. In order to make the function invertible, the domain needs to be restricted either before the maximum point of the graph to $0 < x \leq 1.81$ or after the maximum point of the graph to $1.81 \leq x < 5$.

5 Quadratic functions can be easily written in vertex form, which then allows the input variable *x* to be easily isolated. Cubic function usually cannot be written in a similar form and thus the input variable *x* cannot be isolated.

6 Since we need to isolate the input variable *x*, in order to determine the inverse function and since this process is usually difficult or even impossible with a cubic function, the inverse of a cubic function is difficult to be found.

7 $x = 0, x = 5, x = 6$

8 $x = 0$ and $x = 5$ represent the upper and lower limits for the size of x. The model does not apply for values of x that are larger than 5 would give you a negative value for the width of the box.

9 The local maximum value is 96.77 when $x = 1.81$ and the local minimum value is -5.51 when $x = 5.52$.

10 It is not possible for the volume to have a negative value. The smallest value for the volume of the box is 0 cm^3 when x = 0 cm or x = 5 cm.

11 Given $0 \leq x \leq 5$, you could use this model to predict the volume of the box. However, outside these values the model is not valid.

12 **Conceptual:** Using your answer to question **11**, do you think that, in general, cubic models could be used to predict information about real-life situations?

Answer: Cubic functions can be used to model real-life data, while displaying both a minimum and a maximum value.

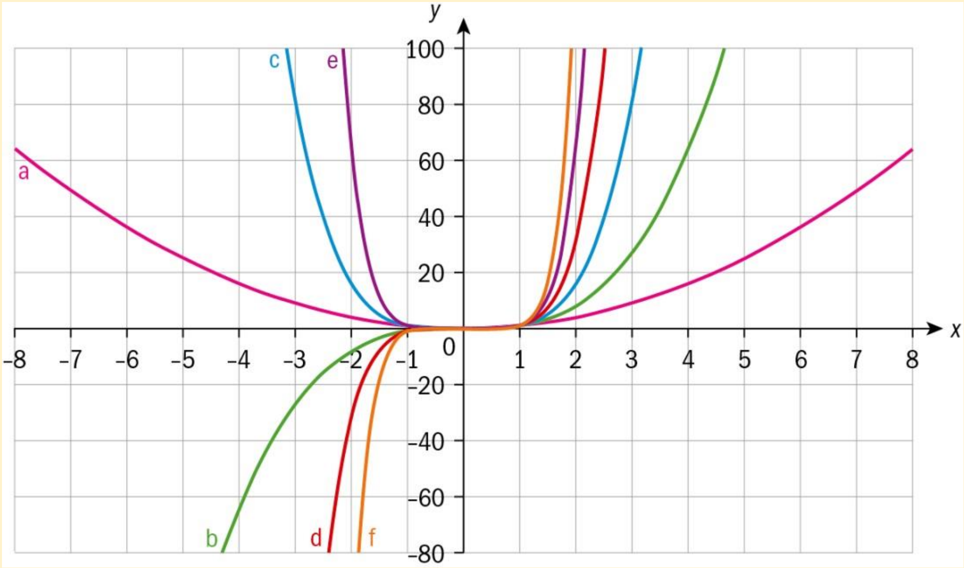
6.4 Power functions, direct and inverse variation and models

Investigation 4

Conceptual understanding:

The graph of positive power functions goes through the origin and displays features such as symmetry while the graph of negative power functions displays features such as symmetry and asymptotic behavior.

1 a-f



2 Odd and even power functions.

3 Even power functions are similar to parabolas. Odd are partly similar and partly reflected.

4 Odd power functions are similar to cubic functions.

5 Even power functions have a minimum point. Odd power functions have no maximum or minimum points and instead have a point of inflexion.

6 Even power functions are symmetric about the *y*-axis. Odd power functions are symmetric about the origin.

7 They become narrower.

8 **Conceptual:** What are the distinguishing geometrical features of positive and negative power functions?

Answer: The graph of positive power functions goes through the origin and displays features such as symmetry while the graph of negative power functions displays features such as symmetry and asymptotic behavior.

Investigation 5

Conceptual understanding:

The inverse proportional function can be used to model real-life data with asymptotic behavior in which, as the dependent variable increases, the independent variable decreases.

1 The greater the speed, the less time will be required to drive to school.

2 The time will now become half of what it normally is.

3 The time will now become double of what it normally is.

4 **Factual:** How could you describe the variation between speed and time?

Answer: The variation between speed and time is inversely proportional. That means that as one value increases the other decreases and vice versa.

5 **Conceptual:** How would you describe real situations that model asymptotic behaviour?

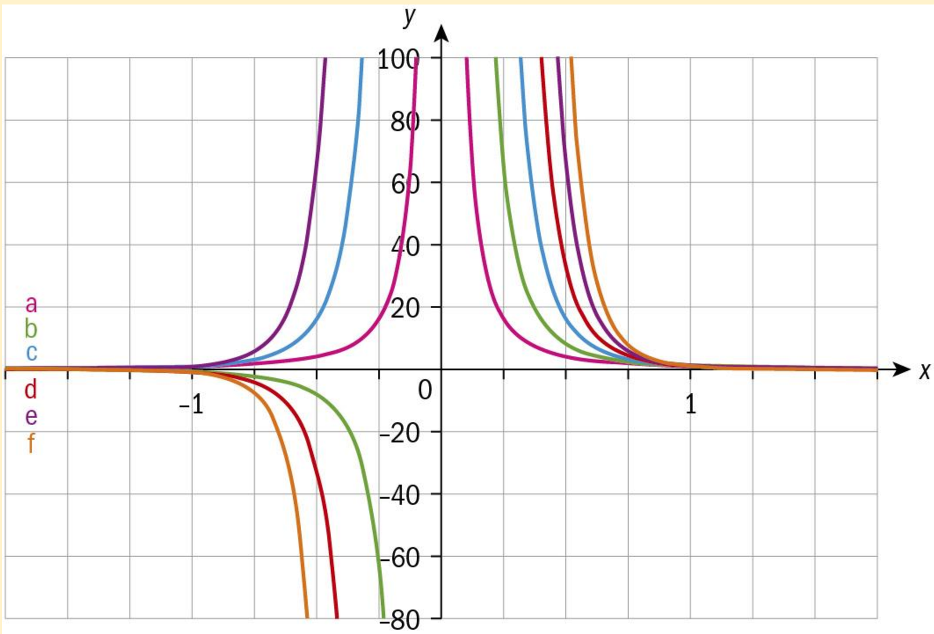
Answer: The inverse proportional function can be used to model real-life data with asymptotic behavior in which, as the dependent variable increases, the independent variable decreases.

Investigation 6

Conceptual understanding:

The inverse proportional function displays features such as symmetry and asymptotes, a vertical and a horizontal line towards which the graph approaches infinitesimally closely, without ever reaching it.

1 a-f



2 They never go through the *y*-axis and they tend to become horizontal for large positive and negative values of *x*.

3 Even negative are entirely above the *x*-axis, whereas odd negative are both above and below the *x*-axis.

4 Even power functions are symmetric about the *y*-axis. Odd power functions are symmetric about the origin.

5 a As *x* becomes very large, the value of *y* becomes very small, asymptotically tending towards 0.

b As *x* approaches 0 from the positive side, the *y* values tend to become very large, asymptotically tending towards ∞.

c As *x* approaches 0 from the negative side, the *y* values tend to become very large, asymptotically tending towards ±∞. Positive for even functions and negative for odd functions.

6 They become narrower.

7 **Conceptual:** What are the distinguishing geometrical features of inverse variation functions?

Answer: The inverse proportional function displays features such as symmetry and asymptotes, a vertical and a horizontal line towards which the graph approaches infinitesimally closely, without ever reaching it.

TOK

Some questions for the development of a discussion.

- What makes something a language?

- Can you communicate in mathematical symbols?
- Could you communicate to people, who speak a different language that you do not understand, by using mathematics?
- What would Godel say? What would Hilbert say?

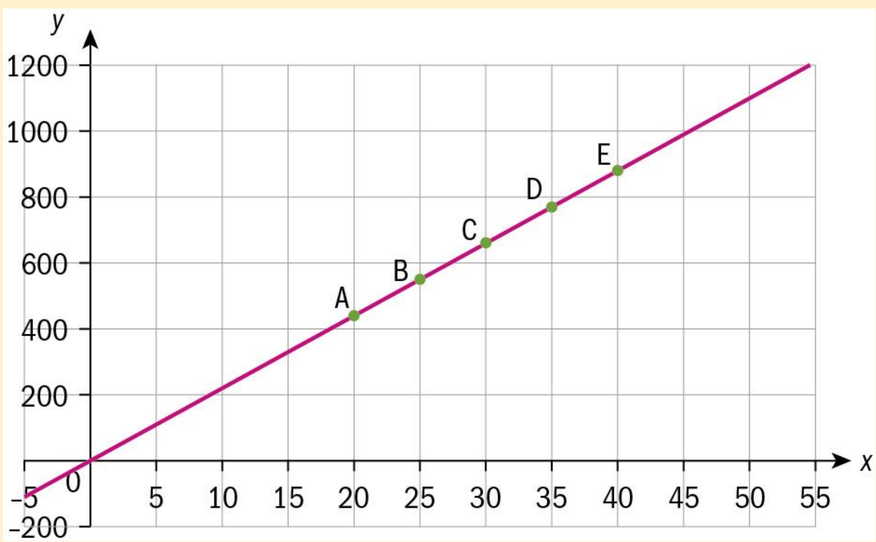
Investigation 7

Conceptual understanding:

The inverse proportional function can be used to model real-life data with asymptotic behavior, in which as the dependent variable increases, the independent variable decreases.

1

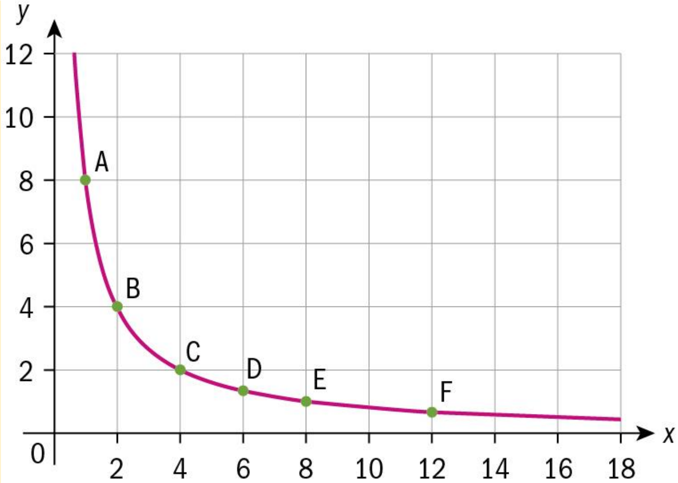
Number of hours	20	25	30	35	40
Pay (€)	440	550	660	770	880



A worker's pay varies directly with the number of hours worked.

2

Number of people	1	2	4	6	8	12
Number of hours	8	4	2	1.33	1	0.67



Factual: How do the number of hours to complete the work vary with the number of men available?

Answer: The number of hours to complete the work varies inversely with the number of men available.

3 **Conceptual:** For problems which involve direct and inverse variation, how does understanding the physical problem help you to choose the correct mathematical function to model the problem with?

Answer: The inverse proportional function can be used to model real-life data with asymptotic behavior, in which as the dependent variable increases, the independent variable decreases.

Investigation 8

Conceptual understanding:

Asymptotes occur when a value leads to division by zero.

1 The value of x that makes the denominator equal to zero is $x = 0$.

2 The line $x = 0$.

3 The value of x that makes the denominator equal to zero is $x = 2$.

4 The line $x = 2$.

5 Translation 2 units to the right.

6 The vertical asymptote translates 2 units to the right as well.

7 The function tends towards $y = 3$.

8 The line $y = 3$.

9 Translation 3 units up.

10 The horizontal asymptote translates 3 units up as well.

11 Horizontal translation h units and vertical translation k units.

12 $x = h$, $y = k$.

13 **Conceptual:** How do you identify an asymptote?

The answer is the TU: Asymptotes occur when a value leads to division by zero.

Developing inquiry skills

How many different trajectories would there be if the ball were to follow a straight line from the hands of the player to the hoop?

Answer: There is only a single straight line that goes through two given points.

How many different trajectories are there in the case that the ball follows a curved line from the hands of the player to the hoop, as in the drawing here?

Answer: There are infinite curved trajectories joining two different points.

How would the initial angle at which the ball leaves the hands of the player affect the angle at which it arrives at the hoop?

Answer: Since quadratic functions display symmetry, a trajectory starting off at a greater angle to the horizontal will arrive at the hoop at a greater angle to the horizontal.

How would this angle increase the chance of the player making the basket?

Answer: As the angle that the ball approaches the hoop increases, the area that it "effectively views" changes.

Knowing the maximum height of the ball, could we determine a unique model for the trajectory it follows?

Answer: Yes, there is a unique quadratic function that goes through three given points.

How does the height up to which the ball goes increase the chance of the player making the basket?

Answer: The higher the ball goes, the bigger the area that it "effectively views" changes. On the other hand, to send the ball higher, more force is required, making the precision to shoot the ball decrease.

How can we determine the optimum path for a basketball player to score a basket?

Answer: Determine the quadratic function that comes to the hoop as vertically as possible, requiring an amount of energy that will not affect precision.

What else would we need to know to answer this question?

Answer: How extra force to send the ball higher affects precision.

What assumptions have you made in the process?

Answer: That the path is purely quadratic.

Hanging around!

Approaches to Learning: Thinking Skills: Create, Generating, Planning, Producing

Exploration Criteria: Presentation (A) Personal engagement (C) Reflection (D)

IB Topic: Quadratic Modelling, Using technology

This task introduces students to the possibility of importing images into graphing packages in order to find a curve to best fit the object in the image. This is something that a lot of students do or could be advised to do in their explorations.

Students will find a curve and reflect on why the curve may not fit the image exactly which will lead them to consider the choice of their curve.

The task therefore encourages personal engagement (Criterion C) by considering using actual data provided by the student and using technology creatively to fit a model. The use of the graph and the equation to represent the model will add to Criterion A: Presentation. A consideration of the suitable fit (or otherwise) of the model will lead to critical reflection for Criterion D: Reflection.

Investigate

For this task, students will need:

- A piece of rope or chain
- Access to a graphing package such as Desmos (which is used in these notes) or Autograph, Geogebra, Loggerpro, Geometer's sketchpad, etc.

If a piece of rope or chain is not available, then students could use a picture of a hanging rope or chain from the internet.

Students will probably suggest that the shape is a parabola (quadratic curve), since they have studied this in chapter **3**. They could also suggest a cosine curve if this has been studied.

$y = ax^2 + bx + c$ or $y = \cos x$

Students could test this by trying to fit a curve that is quadratic (or cosine).

Import the curve into a graphing package

The aim here is to use a graphing package to try to find the equation of the quadratic that best fits the shape of the hanging chain in the form $y = ax^2 + bx + c$.

When taking their photograph, students should realise that it is important to get a clear image with the camera looking staright at the rope/chain rather than from above/below or from the side.

Make sure that students follow the instructions correctly to import their image into the graphing package Desmos, or equivalent.

For example, to import the image into the Desmos graphing package, the steps are:

Open Desmos

Click on the  on the top left

Click on 'image' and locate your image to import.

The image should appear in the graphing screen.

Fit an equation to three points on the curve

It doesn't really matter what three points students select, but different points may result in slightly different equations.

Two points would not be sufficient as there is not a unique quadratic curve going through two points.

For extension, you could also ask:

- Could you use more than three points? What difference would this make?

You could use as many points as you want. Three is the minimum required to fit a unique parabola. More than three points may not produce an R-squared value of 1, but may produce a curve which more closely fits the shape of the image.

For students who need further guidance on how to enter their three chosen points into a table, for example, the points (−2.9, 0), (−0.1, −4.7) and (3, 1) could be used.

Make sure that students follow the instructions correctly for the graphing package Desmos, or equivalent.

The steps for the Desmos graphing package are:

Type $y_1 \sim ax_1^2 + bx_1 + c$

If appropriate, you could tell students that the ~ symbol (called Tilde) means that Desmos will provide the best fit equation of the provided form.

Explain to students that y_1 and x_1 refer to the headings on the table.

The values of a, b and c are given under the equation. These are the values of the coefficients of the quadratic function that best fit the points provided.

For example, for the coordinates used above, the best fit equation is $y = 0.596x^2 + 0.110x - 4.69$, to three significant figures.

Desmos also provides the R^2 (or R-squared) value. For the example used, this value is 1.

If appropriate for extension work, or if students ask what it is, you could introduce the coefficient of determination, R^2.

R^2, also known as the coefficient of determination, is a statistical measure of how close the data are to the fitted curve. A value of 1 means that the curve passes through the 3 points exactly.

You could ask:

- Why is $R^2 = 1$ here?
- Try using four or more points. Does the value of R^2change?

$R^2 = 1$ because Desmos has provided a curve that exactly fits through the three points. This may not be the case if more than three points are used because the curve may no longer fit exactly through all the points.

Test the fit of your curve

No, the curve does not fit exactly.

It does not overlap entirely with the curve, but it is close.

Possible reasons:

The choice of curve might not be correct.

Inaccuracy in selecting and marking points – for example, if the rope or chain is thick and the points in the middle of the rope/chain can’t easily be selected.

Could be a poor-quality image, have a shadow that distorts the picture or taken from a difficult angle and not straight on to the object.

You could also discuss how moving, expanding or contracting the image would affect the equation. The resulting equation students would be working out would be different depending on the image, location and size. However, if the distance between the two points from which the rope is hanging is known, students could work out the scale of the graph.

If students were going to consider areas or geometrical findings based on their curve, they would need to have a sense of scale. They could find this by knowing, for example, the physical distance between the two points where the rope is hooked.

For extension, students could try following the same process and fitting a cosine curve.

They could then investigate the limitations of this approach to finding a curve of best fit.

The catenary and the parabola are two different curves. They look generally quite similar in that they are symmetrical and U-shaped, going up infinitely on either side of a minimum. However they do have a different shape.

A catenary is slightly "flatter" at the bottom and it rises faster than a parabola for large values of x.

For extension work, you could ask:

- What else could you use this process to model?
- What could the resulting equations be used to find?

It would be possible to import any photo like this and students could then fit any curve of which they know the general form of the equation.

One of the key points from this task is that it is important to find the correct type of curve for a particular given situation − this may involve doing some research first!

Once students have an equation it would be possible to perhaps find the area under a curve or a volume of revolution based on the curve (see chapter **11**).

Students may be able to make predictions for the position of other points on the curve that perhaps cannot be seen in a photo/diagram.

It may be possible to find the highest or lowest point of a travelling object or where that object might end up/land.

For extension work, you could also ask:

- What is the minimum number of points you would need to find a cubic model?
- Is it different from a quadratic? Why?

A cubic model would require four points to fit a unique curve as a general cubic curve $y = ax^3 + bx^2 + cx + d$ has four parameters.

In fact a polynomial of degree n would require $n + 1$ points.

Extension

These are just examples that may be modelled using quadratics – students often include many other types of functions that they model – the important point is that they can justify the choice of the model that they select or can suggest why the model does not fit exactly if it doesn’t.

For additional interest: Watch this from Matt Parker (Stand Up Mathematician!) – *Sydney: The Unsuccessful Hunt for Parabola*.

7 Modelling rates of change: exponential and logarithmic functions

Essential understandings

Models are depictions of real-life events using expressions, equations or graphs while a function is defined as a relation or expression involving one or more variables. Creating different representations of functions to model the relationships between variables, visually and symbolically as graphs, equations and tables represents different ways to communicate mathematical ideas.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Different representations of functions, symbolically and visually as graphs, equations and tables provide different ways to communicate mathematical relationships.
- The parameters in a function or equation correspond to geometrical features of a graph and can represent physical quantities in spatial dimensions.
- Moving between different forms to represent functions allows for deeper understanding and provides different approaches to problem solving.
- Our spatial frame of reference affects the visible part of a function and by changing this "window" can show more or less of the function to best suit our needs.
- Different representations of functions, symbolically and visually as graphs, equations and tables provide different ways to communicate mathematical relationships.
- Changing the parameters of a trigonometric function changes the position, orientation and shape of the corresponding graph.
- Different representations facilitate modelling and interpretation of physical, social, economic and mathematical phenomena, which support solving real-life problems.
- Technology plays a key role in allowing humans to represent the real world as a model and to quantify the appropriateness of the model.
- Extending results from a specific case to a general form and making connections between related functions allows us to better understand physical phenomena.
- Generalization provides an insight into variation and allows us to access ideas such as half-life and scaling logarithmically to adapt theoretical models and solve complex real-life problems.
- Considering the reasonableness and validity of results helps us to make informed, unbiased decisions.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	Investigation
In an arithmetic sequence, to get from one term to the next, you add (or subtract) a constant number. Terms in a geometric sequence increase (or decrease) by the common ratio, which may be positive, negative, or fractional. Recognizing patterns and identifying the first term and common ratio will help to find the general term of an arithmetic sequence.	Investigation 1
A geometric sequence can model real-life situations that reflect a constant percentage change. Geometric sequences with common ratio greater than 1 show growth, whereas geometric sequences with fractional common ratio show decay and tend to become infinitely small (or infinitesimal).	Investigation 2

A geometric series will have a finite sum whenever the modulus of the common ratio is less than one.	Investigation 4
Compound interest can be modelled as a geometric growth with common ratio $1+\frac{p}{100}$ where p is the percentage increase per time interval.	Investigation 5
An annuity represents situations with regular constant investments for a fixed time period and compounded growth with the same interest rate.	Investigation 6
If a quantity increases continuously at a constant rate *r* after *t* time units it will have increased by a factor of e^{rt}.	Investigation 11
Exponential equations with different bases may have no integer or exact rational solutions, in which case these solutions can be determined using technology.	Investigation 12
Logarithm laws give a means of changing multiplicative processes into additive processes and this can provide the means to find inverses of exponential functions.	Investigation 13
Logarithms represent inverse functions of the exponential functions and display the same relationship in different ways.	Investigation 14

Syllabus sections covered in this chapter:

- SL1.3*
- SL1.4*
- SL1.5*
- SL1.7
- SL2.2*
- SL2.3^
- SL2.4
- SL2.5
- SL2.6
- AHL1.9
- AHL1.11
- AHL2.10
- AHL2.7
- AHL2.8

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 288: Modelling rates of change: exponential and logarithmic functions	Page 293: Example 2 Page 297: Example 5 Page 304: Example 7	Page 293: Example 2 Page 297: Example 5 Page 299: Example 6 Page 304: Example 7 Page 305: Example 8 Page 306: Example 9 Page 308: Example 10 Page 315: Example 12 Page 316: Example 14	Pages 300, 309, 317, 327,330

Assessment opportunities

End of chapter summary	Chapter review	Exam-style questions
Page 331	Page 332	Page 334

7.1 Geometric sequences and series

Investigation 1

Conceptual understanding:

In an arithmetic sequence, to get from one term to the next, you add (or subtract) a constant number.

Terms in a geometric sequence increase (or decrease) by the common ratio, which may be positive, negative, or fractional.

Recognizing patterns and identifying the first term and common ratio will help to find the general term of an arithmetic sequence.

1 **a** Multiplication by 10.

b Division by 2, or multiplication by $\frac{1}{2}$.

c Multiplication by $\frac{4}{5}$.

d To get from one term to the next, you multiply (or divide) by a certain constant number.

Conceptual: Why aren't these sequences arithmetic?

Answer is the TU: In an arithmetic sequence, to get from one term to the next, you add (or subtract) a constant number.

2

	1st term	2nd term	3rd term	4th term	5th term	6th term
(a)	1	10	100	**1000**	**10000**	**100000**
(b)	64	32	16	**8**	**4**	**2**
(c)	40000	32000	25600	**20480**	**16384**	**13107.2**
(d)	40	-20	10	**-5**	**2.5**	**-1.25**

a increasing **b** decreasing **c** decreasing **d** oscillating

Conceptual: When do the terms of a geometric sequence increase? When do they decrease? When do they oscillate?

Answer is the TU: Terms in a geometric sequence increase (or decrease) by the common ratio, which may be positive, negative, or fractional.

3 **a** The second term is one term after the first. The power to which you raise the common ratio to go from the first term to the second is one.

b The third term is two terms after the first. The power to which you raise the common ratio to go from the first term to the third is two.

c The eighth term is seven terms after the first. The power to which you need to raise the common ratio to go from the fifth term to the eight is seven. $u_8 = u_1 r^7$

d The eighth term is three terms after the fifth. The power to which you need to raise the common ratio to go from the fifth term to the eighth is three. $u_8 = u_5 r^3$

e The nth term is $(n-1)$ terms after the first. The power to which you need to raise the common ratio to go from the first term to the nth is $(n-1)$. $u_n = u_1 r^{n-1}$

Conceptual: How can you find the general term of a geometric sequence?

Answer is the TU: Recognizing patterns and identifying the first term and common ratio will help to find the general term of an arithmetic sequence.

Investigation 2

Conceptual understanding:

A geometric sequence can model real life situations that reflect a constant percentage change.

Geometric sequences with common ratio greater than 1 show growth, whereas geometric sequences with fractional common ratio show decay and tend to become infinitely small (or infinitesimal).

1 0.6% corresponds to $\frac{0.6}{100} = 0.006$

The increase of the population from the beginning of 2011 to the beginning of 2012 is $\frac{0.6}{100} \times 63.2 = 0.006 \times 63.2$

The popoulation at the beginning of 2012 becomes $63.2 + 63.2 \times 0.006 = 63.2 \times (1 + 0.006) = 63.2 \times 1.006$

2 Since the popoulation keep increasing by 0.6%, we need to keep multiplying by 1.006 to determine each consecutice value. Each time we multiply by 1.006, we find the population one year later. In order to get from the population at the beginning of 2011 to the population at the beginning of 2017, we need to find the population 6 years later, which means that we need to multiply by the common ratio (1.006) 6 times. This means that the value of n needs to be 6

3 $P_n = P_0 \cdot \left(1 + \frac{p}{100}\right)^n$

4 In a geometric series, the formula to determine the n^th^ term is $u_n = u_1 \cdot r^{n-1}$. The index on the common ratio is (n-1) since the number of steps from the first term to the 5^th^ term are (n-1). Ia the population example abvoe the formula is $P_n = P_0 \cdot \left(1 + \frac{p}{100}\right)^n$. The index on the common ration is (n) since the number of steps (years in this case) fron the original population to that n years later is (n).

5 $P_{2025} = 127 \cdot \left(1 - \frac{0.2}{100}\right)^7 = 125.232632511 \simeq 125\,\text{million}$

6 $127 \cdot \left(1 - \frac{0.2}{100}\right)^n < 120 \Rightarrow n > 28.3 \Rightarrow n \geq 29$. The year in which the population will fall below 120 million will be the year 2018+29=2047.

7 $P_n = P_0 \cdot \left(1 - \frac{p}{100}\right)^n$

8 **Conceptual:** What type of sequence models a situation with constant percentage change.

> **Answer is the TU:** A geometric sequence can model real life situations that reflect a constant percentage change.

Conceptual: How do geometric sequences model real-life situations

> **Answer is the TU:** Geometric sequences with common ratio greater than 1 show growth, whereas geometric sequences with fractional common ratio show decay and tend to become infinitely small (or infinitesimal).

TOK

We need proofs in mathematics; first, because we want to be sure that what we do is correct.

Proof in essential shows you whether a statement is true or not.

In maths, unlike any other area of knowledge, you can prove that what we do is perfectly correct, because mathematics is not dependent on most ways of knowing but simply on reason.

As a counterclaim, you might want to have students research the link between proof and intuition linked to Ramanujuan.

Investigation 3

1 $r \times S_5 = r \times \left(u_1 + u_1 r + u_1 r^2 + u_1 r^3 + u_1 r^4\right) = u_1 r + u_1 r^2 + u_1 r^3 + u_1 r^4 + u_1 r^5$

2 a $r \times S_5 - S_5 = \left(u_1 r + u_1 r^2 + u_1 r^3 + u_1 r^4 + u_1 r^5\right) - \left(u_1 + u_1 r + u_1 r^2 + u_1 r^3 + u_1 r^4\right) = u_1 r^5 - u_1$

b $\Rightarrow S_5 \times (r-1) = u_1 \times \left(r^5 - 1\right) \Rightarrow S_5 = \dfrac{u_1 \times \left(r^5 - 1\right)}{r-1}$

3 Changing the sign of both the numerator and denominator of a fraction keeps the fraction unchanged.

$$S_n = \frac{u_1 \times \left(r^n - 1\right)}{r-1} = \frac{-u_1 \times \left(r^n - 1\right)}{-(r-1)} = \frac{u_1 \times \left(1 - r^n\right)}{1-r}$$

The reason that this version of the formula might be easier to use for r<1 is that the denominator will always be a positive number.

4 $S_{64} = \dfrac{1 \times \left(2^{64} - 1\right)}{2-1} = 18446744073709551615$ grains of rice *or* 1.84×10^{19}

5 $\dfrac{1.84 \times 10^{19}}{1.2 \times 10^{16}} \simeq 1537$

This value is equivalent to the production of rice for the next 1537 years! In other words until the year 3555!

TOK

You will see the use of several alphabets in mathematical notation (e.g. the use of capital sigma for the sum). One point of view is that mathematics is not only a language but is the only language shared by humans around the world. For example, pi is 3.14159… regardless of what culture, language, nationality or religion you have.

A counterclaim might be whether or not we can communicate our ideas without the use of another spoken tongue.

Investigation 4

Conceptual understanding:

A geometric series will have a finite sum whenever the modulus of the common ratio is less than one.

1 Using the standard formula we obtain $u_{10} = 2.36$, $u_{50} = 107$ and $u_{100} = 12528$

2 $u_{10} = 0.387$, $u_{50} = 0.00572$ and $u_{100} = 0.000295$

3 The first is an increasing function the second a decreasing function. This is because powers of a number with a modulus greater than 1 will increase exponentially and those with a modulus less than 1 will decrease exponentially.

4 The limit is 0, exponential decrease.

5 When $r > 1$ the terms will tend to infinity. For $r < -1$ the terms will oscillate positive and negative but will have increasing magnitude.

6 **a** $u_n = \frac{1}{2^n}$ **b** $\frac{1}{2}$ **c** S_n is the total of the shaded area

d The limit of S_n will be the area of the paper so 1 m^2

e Substituting in the values we obtain $S_n = \dfrac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}$ as $\left(\frac{1}{2}\right)^n \to 0$ as $n \to \infty$ the sum tends to 1.

f From above $\lim\limits_{n\to\infty} r^n = 0$ is when $-1 < r < 1$. Hence the limit of S_n is $S_\infty = \dfrac{u_1}{1-r}$

7 **a** $r = 0.1 \Rightarrow S_\infty = \dfrac{1}{1-0.1} = 1.11\ldots$ **b** $r = -\frac{1}{2} \Rightarrow S_\infty = \dfrac{8}{1-\left(-\frac{1}{2}\right)} = \dfrac{16}{3}$

8 **Conceptual:** When does a geometric series have a finite sum?

Answer: A geometric series will have a finite sum whenever the modulus of the common ratio is less than one.

TOK

Consider for instance that a finite area can be bounded by an infinite perimeter.

This is an opportunity to study the von Koch snowflake.

7.2 Financial applications of geometric sequences and series

Investigation 5

Conceptual understanding:

Compound interest can be modelled as a geometric growth with common ratio $1+\frac{p}{100}$ where p is the percentage increase per time interval.

The more frequently interest is paid the higher the return.

1 5% of 10 000 is €500

2 10 000 + 500 = €10 500

3 $10000(1.05)^n$

4 €16 289

5 **Conceptual:** How can you model compound interest?

Answer is the TU: Compound interest can be modelled as a geometric growth with common ratio $1+\frac{p}{100}$ where p is the percentage increase per time interval.

6 12

7 Interest per month is $\frac{5}{12}$%.

i In the first year this is paid 12 times so the amount in the account will be

$10000\left(1+\frac{5}{1200}\right)^{12}$ = €10 512

ii The interest will be paid 120 times, hence the amount in the account will be

$10000\left(1+\frac{5}{1200}\right)^{120}$ = €16 470

8 Actual percentage gain = $\frac{512}{10000}\times 100 = 5.12\%$

9 The more frequently interest is paid the higher the return.

TOK

Students could research the reason as to why we charge interest on a loan and compare this with the perspectives in other societies such as where money in Islam is not regarded as an asset from which it is ethically permissible to earn a direct return. The Qur'an (2:279) sees interest as inequitable, as implied by the word "zulm" in Arabic which translates as oppression, exploitation, and the opposite of justice. There is no real loaning in Islam since lenders achieve ownership in the estates that they finance.

This allows students to view the perspectives of other societies and decide to what extent they agree with the charging of interest.

Investigation 6

Conceptual understanding:

An annuity represents situations with regular constant investments for a fixed time period and compounded growth with the same interest rate.

1 $V_{10} = 3000 \cdot \left(1 + \frac{6}{100}\right)^{10} = 5372.54$

2 $V_{10-1} = 3000 \cdot \left(1 + \frac{6}{100}\right)^{9} = 5068.44$

3 $\dfrac{3000 \cdot \left(1 + \frac{6}{100}\right)^{10}}{3000 \cdot \left(1 + \frac{6}{100}\right)^{9}} = 1.06$

4 Geometric series with common ratio 1.06

5 $S_{10} = \dfrac{3000 \cdot \left(1.06^{10} - 1\right)}{1.06 - 1} = 39542.38$

Conceptual: How does an annuity represent compounded growth?

Answer is the TU: An annuity represents situations with regular constant investments for a fixed time period and compounded growth with the same interest rate.

7.3 Exponential functions and models

TOK

Consider the ethical perceptions of borrowing and lending money.

The discussion on the integrity of moneylenders has been around for thousands of years, there is even a passage in the Bible referring to this. The German term for debt is “schuld” which is the same word as for guilt, blame and fault.

You might want to ask if lending money is ever ethical, and a great scenario to pose to students is “What then could an ethical lender look like in an ideal world?”

Investigation 9

Conceptual understandings:

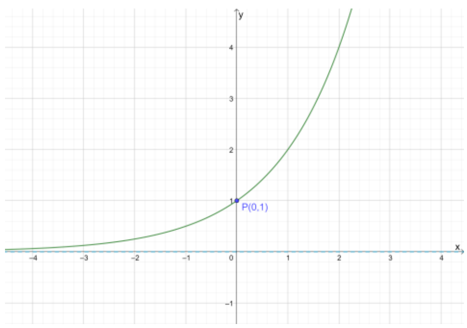
Different representations of functions, symbolically and visually as graphs, equations and tables provide different ways to communicate mathematical ideas. The parameters of these functions correspond to geometrical features of their graph and can represent physical quantities in the real world.

Extending results from a specific case to a general form and making connections between related functions allows it to better understand physical phenomena.

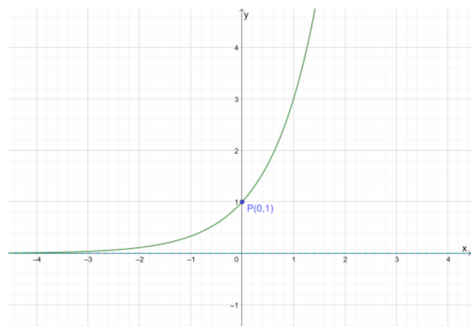
The parameters of exponential functions transform the function by altering the asymptotes, domain/range , the rate of growth or decay and its intercepts and these functions can model real-life situations.

1

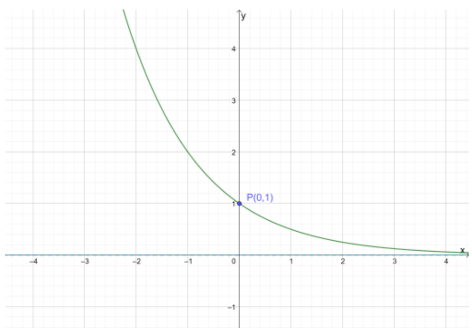
a



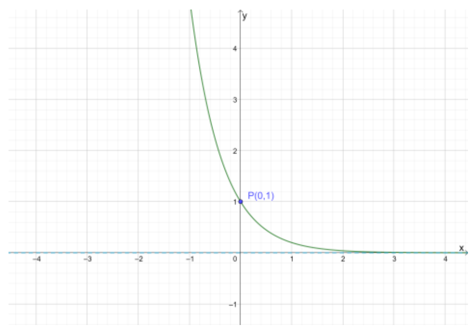
b



c



d



e

2 They don’t have any turning points.

3 The base is between 0 and 1 or the exponent is negative.

4 Exponential functions have their variable in the exponent, while power functions have the variable in its base.

5 They are either constantly increasing or constantly decreasing. Also, they either tend to become horizontal or start from being horizontal.

6 Apart from the fact that some are increasing, and some are decreasing, others are more, and others are less, steep.

7 Some graphs tend to become horizontal, while others start from being horizontal.

8 They don’t have any zeros since any positive number to any power can never be equal to zero.

9 No, since negative numbers do not have roots. And thus we could not raise the base to any rational power.

10 **Conceptual:** Describe the main features of the graphs of all exponential functions of the form $f(x) = a^x, a > 0$.

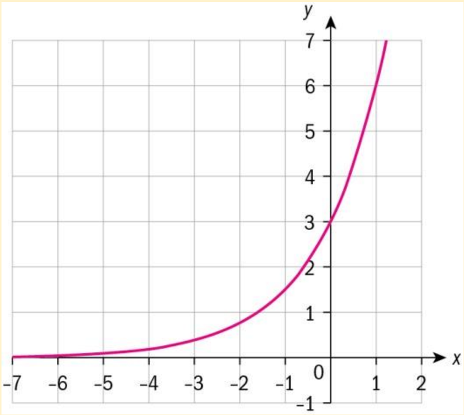
Investigation 10

Conceptual understanding:

The parameters of the exponential function affect the graph of the function by altering the asymptotes, domain/range, the rate of growth or decay and its intercepts, and these functions can model real-life situations.

1 **a** **i** A stretch parallel to the y-axis scale factor 3.

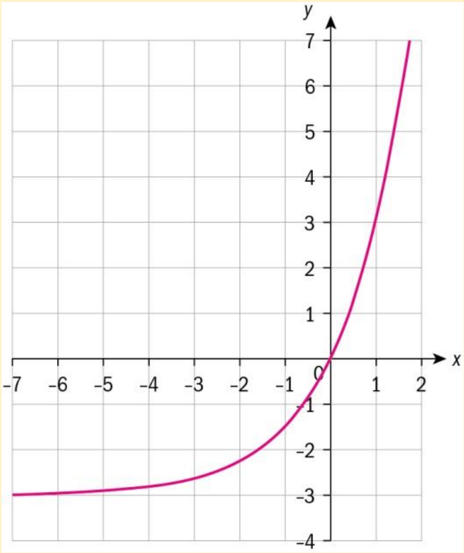
ii



iii $y = 0$ **iv** 3

b A translation of 3 down:

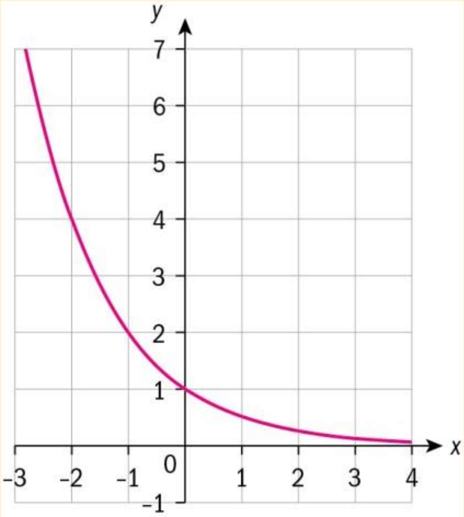
ii



iii $y = -3$ **iv** 0

c **i** Reflect in the y-axis

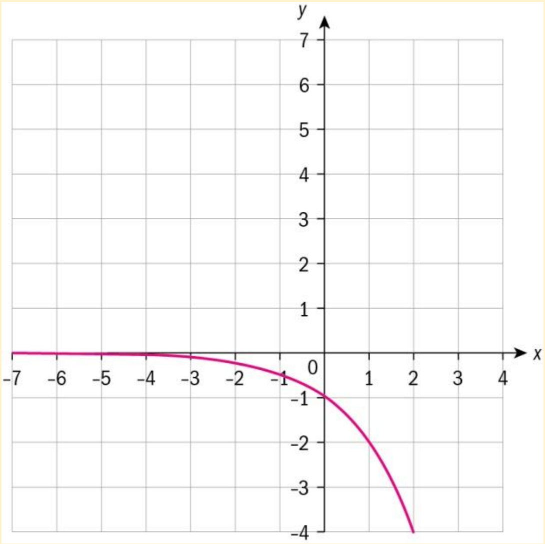
ii



iii $y = 0$ **iv** 1

d **i** A reflection in the x-axis

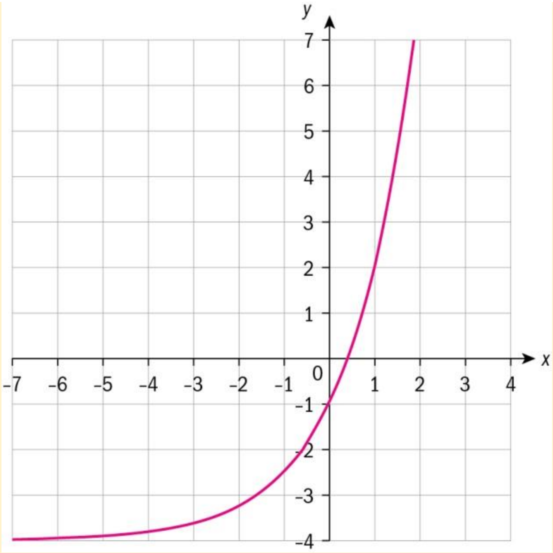
ii



iii $y = 0$ **iv** -1

e **i** A stretch parallel to the y-axis scale factor 3, followed by a translation of 4 down.

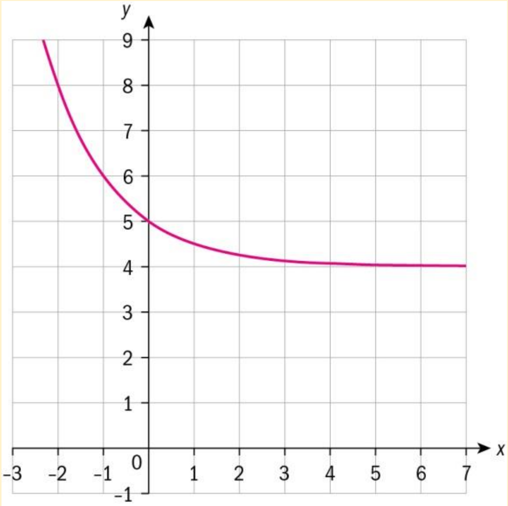
ii



iii $y = -4$ **iv** -1

f **i** A reflection in the *y*-axis followed by a translation of 4 up.

ii



iii $y = 4$ **iv** 5

2 The asymptote is $y = c$, *y*-intercept is $k + c$. For $a > 1$ the curve is increasing exponentially.

3 The asymptote and the y intercept are still $y = c$ and $k + c$ respectively but the curve is now decreasing exponentially.

4 **a** $f(x) = 3^2 \times 3^x + 5 = 9 \times 3^x + 5$ **b** $f(x) = \left(3^2\right)^x + 7 = 9^x + 5$

5 $f(x) = k_1 a_1^{bx+d} + c = k_1 a_1^d \left(a_1^b\right)^x + c$

6 **Conceptual:** How do the parameters of the exponential function affect the graph of the function?

Answer: The parameters of the exponential function affect the graph of the function by altering the asymptotes, domain/range, the rate of growth or decay and its intercepts, and these functions can model real-life situations.

TOK

How does exponential growth in mathematics differ from its use in English?

Is language an inadequate vehicle for expressing everything we can experience and think?

Investigation 11

Conceptual understanding:

If a quantity increases continuously at a constant rate r after t time units it will have increased by a factor of e^{rt} .

1 **a** $2P$ **b** $2.61P$ **c** $2.69P$ **d** $2.71P$

2 The values are increasing and seem to be tending to a limit.

3 The formula is $P\left(1+\frac{r}{100k}\right)^{kn}$ and $n=1,\ r=100$ hence the formula is $P\left(1+\frac{1}{k}\right)^{k}$

4 Using the table function on a GDC or a spreadsheet, the limit e=2.718… can be found.

5 **a** $P\left(1+\frac{100r}{100k}\right)^{k}=P\left(1+\frac{r}{k}\right)^{k}$

b Let $\frac{k}{r}=m$: $\left(1+\frac{1}{\left(\frac{k}{r}\right)}\right)^{k}=\left(1+\frac{1}{m}\right)^{mr}=\left(\left(1+\frac{1}{m}\right)^{m}\right)^{r}$, $\lim_{m\to\infty}\left(\left(1+\frac{1}{m}\right)^{m}\right)^{r}=e^{r}$

6 **Conceptual:** What is the connection between e and the rate of growth of a quantity.

Answer is the TU: If a quantity increases continuously at a constant rate r after t time units it will have increased by a factor of e^{rt}

7.4 Laws of exponents – laws of logarithms

Investigation 12

Conceptual understanding:

Exponential equations with different bases may have no integer or exact rational solutions, in which case these solutions can be determined using technology.

a x is the power to which you need to raise 2 in order to become 8.

b x is the power to which you need to raise 2 in order to become 5. There is no integer or exact rational solution to this equation and so you cannot determine it without the use of technology.

c There are no solutions to this equation.

d x is the power to which you need to raise a in order to become b.

e **Answer is the TU:** Exponential equations with different bases may have no integer or exact rational solutions, in which case these solutions can be determined using technology.

Investigation 13

Conceptual understanding:

Logarithm laws give a means of chanign multiplicative processes into additive processes and this can provide the means to find inverses of exponential functions.

1 $x = 0$; $\log_a 1 = 0$

2 $x = 1$; $\log_a a = 1$

3 $x = n$; $\log_a\left(a^n\right) = n$

4 $\log_a x + \log_a y = \log_a(xy)$

5 $\log_a x - \log_a y = \log_a\left(\dfrac{x}{y}\right)$

6 $\log_a(x^n) = n \cdot \log_a x$

TOK

How does exponential growth in mathematics differ from its use in English?

Is language an inadequate vehicle for expressing everything we can experience and think?

Investigation 14

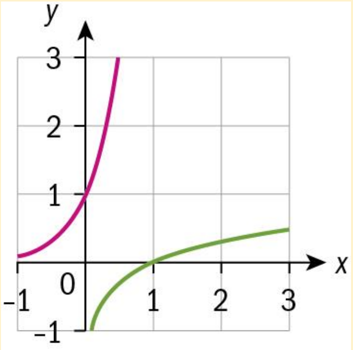
This investigation introduces the graphs of log functions and demonstrates that log and exponential functions are inverse functions.

1 **a** $\log_{10} 10^x = x$ using the log laws or the definition of logs.

$10^{\log_{10} x} = x$; again, this uses the definition of logs, $\log_{10} x$ is the power you need to raise 10 by to get *x.*

b Hence they are the inverse of each other.

2 As they are inverses, $y = \log x$ is the reflection of $y = 10^x$ in the line $y = x$.

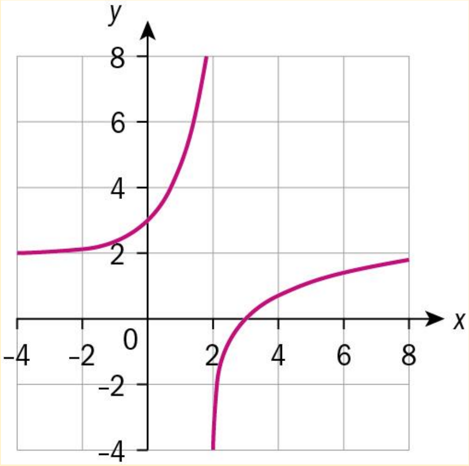


3 The domain is $x > 0$ and the range is $\mathbb{R}$ the reverse of $y = 10^x$.

4 **a** $y = e^x + 2$

$x = e^y + 2 \Rightarrow e^y = x - 2 \Rightarrow y = \ln(x-2)$, $h^{-1}(x) = \ln(x-2)$

b



5 The domain of h^{-1} is $x > 2$ and the range is $\mathbb{R}$. The domain of h is $\mathbb{R}$ and the range is $h(x) > 2$.

6 **Factual:** What are the domain and range of the function $f(x) = \log_a x$?

Answer: The domain is $x > 0$ and the range is $\mathbb{R}$.

7 **Conceptual:** What is the relationship between the logarithmic function and the exponential function?

Answer: Logarithms represent inverse functions of the exponential functions and display the same relationship in different ways.

TOK

- Are logarithms a natural occurrence or are they a human invention?
- Is mathematics created to solve real world problems?
- Do you need imagination to create new mathematics?
- Does faith have a role to play in the careers of mathematicians?

The natural logarithm appears in physics, biology, sociology, economics and more. Students of physics know that many of the calculations, for example in electrodynamics and quantum mechanics, would be impossible if it were not for the natural logarithm. The universal applicability of the natural logarithm suggests that it is something that exists in the world in which we live and therefore it is a characteristic of the natural world.

7.5 Logistic models

TOK

Some mathematical constants like π, e and the Fibonacci numbers appear consistently in nature. Research where these may be found and consider if they are natural occurrences or are we applying the mathematics that we know to these instances?

A passing fad?

Approaches to Learning: Communication, Research

Exploration Criteria: Mathematical presentation, Reflection (D), Use of mathematics (E)

IB Topic: Exponentials and Logarithms

This task gives further practice to students in finding data and using it to model (in this case exponential functions) and then reflecting on the usefulness of the model for predictions. Students could model by hand or use technology (or both). Students are not penalized for using technology if they can demonstrate understanding of the process that is being used.

At the time of writing, Fortnite is a massive phenomenon and the task here is to see whether this is a passing fad or if the exponential growth continues. Similar discussions could be had regarding 'the latest fad' and if data could be found then this could form the basis of the discussions for the task at the end.

Depending on where this is covered, during the chapter and previously you may wish to work through the calculations of the 'by hand' model and the model using technology. Students can then use this for their own data or, if they are able to, they could then devise these models themselves.

There are opportunities in the task for discussing the importance for consistent notation in an IA as well as Reflection (Criterion D) on the reliability of data that they can find.

Look at the data

The data is taken from the press releases of the developers, Epic Games, but will supposedly be subject to checks and scrutiny by rival companies.

The data is to the nearest million, so is not particularly accurate.

The data is of users but some of these users may play frequently and some may play only once in the time period being measured.

The dates are not very accurate – you assume the data is all released at the same point in the month.

Daily Average Users (DAU) or Monthly Average Users (MAU) or amount of time spent on the game may be more interesting information to collect if it is possible to find.

The growth looks exponential. This could be due to word of mouth with people suggesting to friends, etc, to play the game.

Model the data

The model could be useful in terms of predicting future advertising prices and revenues for the company or for rival companies to consider when a game may be reaching saturation point.

Emphasize the importance of making sure that the variables in the model are clearly defined.

Look at these models (or equivalent using the available software) with students:

By hand	By technology (using TI-NSPIRE)	By technology (using Desmos)
Choose 2 points (t, P) For example, (1,1) and (6,45). Substitute into the equation: $P = a \cdot b^t$ $1 = a \cdot b^1$ $\therefore a \cdot b = 1$ $a = \frac{1}{b}$ and $45 = a \cdot b^6$ So $45 = b^5$ $b = \sqrt[5]{45}$ $b = 2.14$ (3sf) $a = \frac{1}{2.14}$ $a = 0.467$ (3 sf) $\therefore P = 0.487(2.14)^t$ Draw the curve. Note: if you choose 2 different points you would get a different curve.	Enter the data into a table. Label *x* list *t* and *y* list *P*. Menu > Statistics > *A*: Exponential Regression. Select *x* list as *t* and *y* list as *P*. Press OK. Therefore, the equation of the exponential curve that best fits the data according to the GDC is $P = 1.81 \cdot 1.56^t$.	The data can be inputted on a table in Desmos and then a function can be found using regression by inputting the function $y_1 \quad a \cdot b^x$ The equation of the exponential curve that best fits the data according to Desmos will be found. This is $P = 1.81 \cdot 1.56^t$. Make sure that Desmos is in 'log mode' to obtain the same result. If log mode is not ticked then a different equation will be found. $P = 8.65 \cdot 1.27^t$ (this may be an opportunity to discuss/explore residuals. By clicking residuals 'plot' this gives a good visual measure of how accurate the model is).

The 'by hand' model only uses 2 points. The curve will naturally pass through these 2 points but is not necessarily close to any of the other points.

You could also choose a different function of a similar form, say $P = a \cdot b^t + c$, and find parameters *a*, *b* and *c* for the model. This is not a function that is available as an option on the GDC but you could use Desmos, for example, to find it.

If you were to find a model $P = a \cdot b^t + c$ by hand you would require 3 points to find the 3 variables.

You need to calculate how many months it is since July 2107. Substitute a value. The result is likely to be very large and discussion will be around the fact that the game will have reached saturation or a new game will have come along, etc.

Students could research the number of Fortnite players there are in the current month.

You could ask:

- Is Fortnite still a popular game now?

Students could compare this figure with their prediction based on their model.

- How big is the error? What does this tell you about the reliability of your previous model?

This will hopefully support the above.

Plot a new graph with the updated data you have found and try to fit another function to this data.

- Will a modified exponential model be a good fit? If not, what other function would be a better model that could be used to predict the number of users now?

This could be a good opportunity for a discussion around a logistic model that may be more appropriate. This from Khan Academy (https://goo.gl/KMmFbC) is a good summary of what happens when an exponential model is constrained by real life.

Extension

Hopefully there are numerous examples of 'the next big thing'. Good sources will be social media sites, games, technology uptake, etc.

Students should be encouraged to find their own data, display and model it.

This task could be written up as a mini-exploration, perhaps assessed against a smaller number of criteria.

Here is a possible structure for this:

Mini-exploration:

Write a brief exploration on what you find out.

This exploration should be between one and two pages depending on the number of diagrams/graphs that you use.

This is not an exercise in being able to copy from Wikipedia or other websites, but rather to find out relevant information and to rewrite it into an exploration.

Students could be marked against parts of the Criteria of the real Mathematics Exploration:

Criterion A: Presentation (3)

Your writing should be well-organised, coherent, logically developed and easy to follow. It includes an introduction, aim and conclusion.

Criterion B: Mathematical communication (3)

Use appropriate mathematical language and representation and define key terms.

Criterion D: Reflection (3)

You should review, analyse and evaluate your exploration. You should consider the significance of your findings, state possible limitations and/or extensions and make links to different fields and/or areas of mathematics.

Criterion E: Use of mathematics (1)

Demonstrate that you fully understand the mathematics used in your exploration.

TOTAL (10)

8 Modelling periodic phenomena: trigonometric functions and complex numbers

This chapter introduces trigonometric functions and explores their use in modelling periodic phenomena, for example tides, temperatures of populations. It introduces radians as an alternative measure of angles and considers their use in finding areas of arcs and sectors. The second half of the chapter introduces complex numbers and in particular how they can be understood using the Argand diagram. Complex numbers can also be used to describe periodic phenomena and the chapter includes modelling curves using complex numbers using variable moduli and arguments. The final section links the notion of trigonometric functions with complex numbers as it shows how the latter can be used to find the sum of two or more trigonometric functions with equal frequencies, in the particular context of electrical circuits.

Essential understandings

Models are depictions of real-life events using expressions, equations or graphs while a function is defined as a relation or expression involving one or more variables. Creating different representations of functions to model the relationships between variables, visually and symbolically as graphs, equations and tables represents different ways to communicate mathematical ideas.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Different representations of functions, symbolically and visually as graphs, equations and tables provide different ways to communicate mathematical relationships.
- The parameters in a function or equation correspond to geometrical features of a graph and can represent physical quantities in spatial dimensions.
- Moving between different forms to represent functions allows for deeper understanding and provides different approaches to problem solving.
- Our spatial frame of reference affects the visible part of a function and by changing this "window" can show more or less of the function to best suit our needs.
- Changing the parameters of a trigonometric function changes the position, orientation and shape of the corresponding graph.
- Different representations facilitate modelling and interpretation of physical, social, economic and mathematical phenomena, which support solving real-life problems.
- Technology plays a key role in allowing humans to represent the real world as a model and to quantify the appropriateness of the model.
- Extending results from a specific case to a general form and making connections between related functions allows us to better understand physical phenomena.
- Generalization provides an insight into variation and allows us to access ideas such as half-life and scaling logarithmically to adapt theoretical models and solve complex real-life problems.
- Considering the reasonableness and validity of results helps us to make informed, unbiased decisions.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
Radian measure provides a precise measure in finding arc lengths and areas of sectors in a circle as it represents the proportional relationship between the arc length and the radius of the circle.	Investigation 1
Defining the sine of an angle as the vertical distance of a point on a unit circle extends the definition of sine from the first quadrant to all the others and produces a periodic function. Defining the cosine of an angle as the horizontal distance of a point on a unit circle extends the definition of cosine from the first quadrant to all the others and produces a periodic function.	Investigation 2
Inside the sine function, everything is reversed, including the order of operations. Changing the values of c and d shifts the graph horizontally and vertically respectively.	Investigation 3
Complex numbers complete the number systems by allowing unsolvable quadratic equations to be solves with complex solutions occurring in conjugate pairs.	Investigation 4
We use the collection of like terms to add and subtract complex numbers. Simplifying surds will be the same process as i behaves like a square root (of −1).	Investigation 5
If **a** and **b** are the position vectors of A and B, then $\mathbf{a}+\mathbf{b}$ is the position vector of the point P where OAPB forms a parallelogram. Also $\mathbf{a}-\mathbf{b}$ is the position vector of the point Q where OQAB forms a parallelogram. Multiplying two complex numbers together results in geometrically stretching the first complex number z_1 by a factor equal to the magnitude of the second complex number z_2 and then rotating the stretched z_1 counter-clockwise by an angle equal to the argument of z_2 to arrive at the product. Complex numbers can be represented geometrically using vectors to help add or subtract complex numbers.	Investigation 6
Understanding the geometrical behaviour of a power of a complex number can help with more complex calculations.	Investigation 7
The graph of the imaginary part of a complex number corresponds to a sine function and changing the modulus of the complex number alters the amplitude of the function The sum of two sine functions with the same frequency and amplitudes r_1 and r_2 and phase shifts α_1 and α_2 results in a sine function with a modulus and argument transformed from the original sine functions.	Investigation 8

Syllabus sections covered in this chapter:

- SL2.5
- SL2.6
- AHL2.9
- AHL3.8
- AHL2.8
- AHL1.12
- AHL1.13
- AHL3.7

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 338: Modelling periodic phenomena: trigonometric functions and complex numbers	Page 349: Example 4 Page 355: Example 5 Page 362: Example 9 Page 365: Example 10	Page 345: Example 2 Page 349: Example 4 Page 355: Example 5 Page 358: Example 8 Page 362: Example 9 Page 365: Example 10	Pages 342, 350, 356, 363, 366

Assessment opportunities

End of chapter summary	Chapter review	Exam-style questions
Page 366	Page 368	Page 369

Opener

This opening problem seeks to introduce periodic functions to the students and have them think about the main features of such a function.

The bullet points might be addressed as follows:

The model is a good fit in that the period of the function seems to match the periods of the oscillation.

It does not fit the data well at the extremes of the collected data. The function seems to underestimate the actual values.

The function does seem to follow a repeating pattern though the amplitude of the oscillation is not always equal.

If the students are familiar with the sine function they might reference it here. The sine function has a fixed period and amplitude and its value follows periodic increasing and decreasing values about a principal axis, which is similar to the data on display.

Developing inquiry skills

You might expect daylight hours to also vary periodically. The period would be a year rather than the period for tides, which might be monthly. It would also be affected by fewer factors than the tides so would possibly be closer to a sine function. The hours of daylight would also be dependent on the times of both sunrise and sunset. If these do not follow the same pattern then it might introduce distortions into a simple sinusoidal model.

8.1 Measuring angles

TOK

You might want to consider the use of simplicity over accuracy and the value of radians in calculus and physics.

Consider the arbitrary nature of degree measure versus radians as real numbers and the implications of using these two measures on the shape of sinusoidal graphs.

If we can use different measures to represent the same thing, is mathematics such an international language?

Investigation 1

Conceptual understanding:

Radian measure provides a precise measure in finding arc lengths and areas of sectors in a circle as it represents the proportional relationship between the arc length and the radius of the circle.

1 Doubling the angle of the sector will double the length of arc. Hence, if a sector has radius 1 and angle 2 radians, the length of the arc will also be 2.

2 For a semicircle, the angle at the centre is 180°. The length of the arc is half the circumference of the circle so it is $\frac{1}{2}\times 2\times \pi\times 1=\pi$. As a consequence the angle in radians is π radians.

3 **Factual:** What is the length of the arc? Hence state the size of the angle in radians.

Answer: The length of arc if $\theta = 2\pi$ would be the entire circumference which is equal $2\pi r$.

For a general value of θ we'd have $\frac{\theta}{2\pi}$ of the whole circle. Similarly, the area of the sector would be $\frac{\theta}{2\pi}\times \pi r^2 = \frac{1}{2}r^2\theta$.

4 **Factual:** What would be the exact radian equivalence of 90°? 45°? 120°?

Answer: 90° is equivalent to $\frac{\pi}{2}$ radians. 45° is equivalent to $\frac{\pi}{4}$ radians. 120° is equivalent to $\frac{2\pi}{3}$ radians.

5 **a** The circumference of the circle is $2\pi r$ and the fraction required is $\frac{\theta}{360}$ of this.

b For a general value of θ we'd have $\frac{\theta}{2\pi}$ of the whole circle and hence the formula for the length of the arc would be $\frac{\theta}{2\pi}\times 2\pi r = r\theta$

c If one radian subtends an arc length of r then θ radians will subtend an angle of $r\theta$.

6 **a** Area = πr^2 **b** $\frac{\theta}{2\pi}\times \pi r^2 = \frac{1}{2}r^2\theta$

7 **Conceptual:** Why might you choose to use radians rather than degrees when finding the lengths of arcs and areas of sectors?

Answer: Radian measure provides a precise measure in finding arc lengths and areas of sectors in a circle as it represents the proportional relationship between the arc length and the radius of the circle.

8.2 Sinusoidal models

Investigation 2

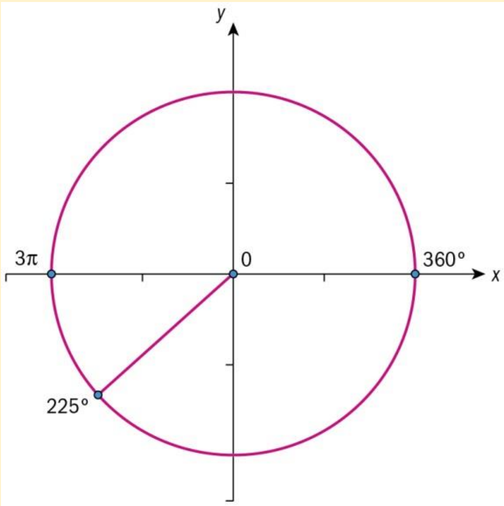
Conceptual understandings:

Defining the sine of an angle as the vertical distance of a point on a unit circle extends the definition of sine from the first quadrant to all the others and produces a periodic function.

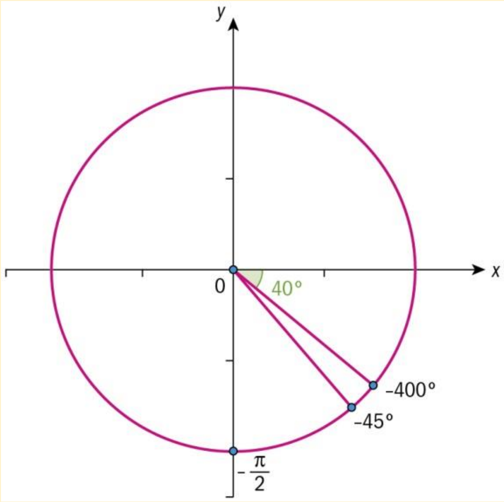
Defining the cosine of an angle as the horizontal distance of a point on a unit circle extends the definition of cosine from the first quadrant to all the others and produces a periodic function.

1 $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$

2



3 Negative angles are measured clockwise from the positive x-axis.



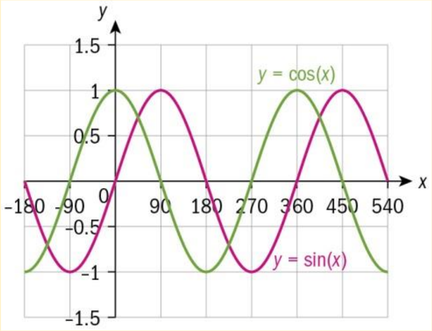
4 **a** The distance from the origin is always equal to 1, so for angles between 0° and 90° we can use the formula $\sin\theta = \dfrac{\text{Opposite}}{\text{Hypotenuese}}$ and $\cos\theta = \dfrac{\text{Adjacent}}{\text{Hypotenuese}}$ to show $\sin\theta = y$ and $\cos\theta = x$.

b $\tan\theta = \dfrac{y}{x} = \dfrac{\sin\theta}{\cos\theta}$

5

θ	-90°	0°	90°	180°	270°	360°	450°
$\sin\theta$	-1	0	1	0	-1	0	1
$\cos\theta$	0	1	0	-1	0	1	0

6



7 **Conceptual:** How do we describe the sine function beyond the first quadrant?

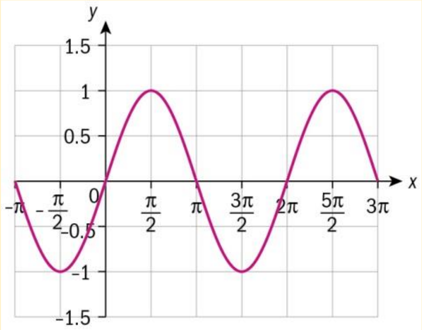
Answer is the TU: Defining the sine of an angle as the vertical distance of a point on a unit circle extends the definition of sine from the first quadrant to all the others and produces a periodic function.

8 **Conceptual:** How do we describe the cosine function beyond the first quadrant?

Answer is the TU: Defining the cosine of an angle as the horizontal distance of a point on a unit circle extends the definition of cosine from the first quadrant to all the others and produces a periodic function.

9 **a** **i** In degrees, as above.

ii



b The similarities will include the same periodic shape and the same amplitude. The differences will include having a different period (wavelength, frequency), differing by a factor of $\dfrac{\pi}{180}$.

10 a The graph will be equivalent to $y=1$

b From the diagram of the unit circle it can be seen by Pythagoras' theorem that the value of x^2+y^2 will always be 1 as the radius of the circle is equal to 1 and hence $\sin^2\theta+\cos^2\theta=1$.

11 a Because the point on the unit circle with angle θ and the point with angle $-\theta$ both have the same x-coordinate and hence the same cosine.

b The point on the unit circle with angle $-\theta$ has the same magnitude of y coordinate as the point with angle $-\theta$ but with different sign. Hence the sine of one is the negative of the sine of the other.

c It can be seen from the graphs that $y=\cos\theta$ is symmetrical in the y-axis and hence $\cos(\theta)=\cos(-\theta)$

$y=\sin\theta$ is symmetrical under a 180° rotation about the origin (it is an odd function) and hence $\sin(\theta)=-\sin(-\theta)$

TOK

Has hearing music ever made you happy or sad? It is said that emotions conveyed by music are a direct result of mathematical relationships in the intervals between the notes.

Mathematics has often been compared with music. For example, in a letter from 1712 to Goldbach, Leibniz remarks that “Music is a hidden arithmetic exercise of the soul, which does not know that it is counting”.

Does this mean that music is mathematical? that mathematics is musical or that both are reflections of a common “truth”?

Scales and base eight might also be explored.

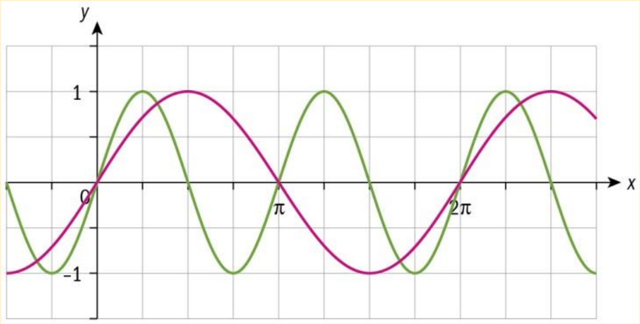
An interesting video can be found at https://youtu.be/ufpatylwPrc

Investigation 3

Conceptual understandings:

Changing the parameters of a trigonometric function may change the position, period and amplitude of the graph.

1 a



b The transformation is a stretch parallel to the x-axis scale factor $\frac{1}{b}$.

c The period of the transformed function will be the period of the original function multiplied by $\frac{1}{b}$ so **i** $2\pi \times \frac{1}{2} = \pi$ **ii** $2\pi \times \frac{1}{4} = \frac{\pi}{2}$

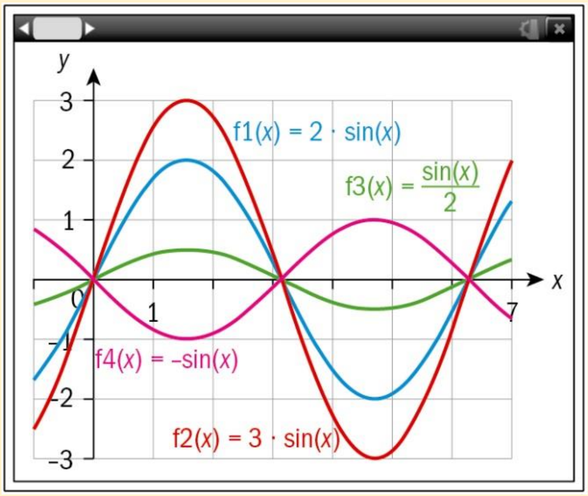
2 **Factual:** How does changing the parameter b change the graph?

Answer: Changing the value of b, stretches the curve parallel to the x-axis with a scale factor $\frac{1}{b}$.

3 The period of $y = \sin bx$ is $\left|\frac{2\pi}{b}\right|$ (or $\left|\frac{360^\circ}{b}\right|$).

4 **a** To map the graph of $y = \sin x$ onto the graph of $y = a \sin x$ you would stretch vertically by a scale factor of a.

b Amplitudes are 2, 3, $\frac{1}{2}$ and 1 respectively.

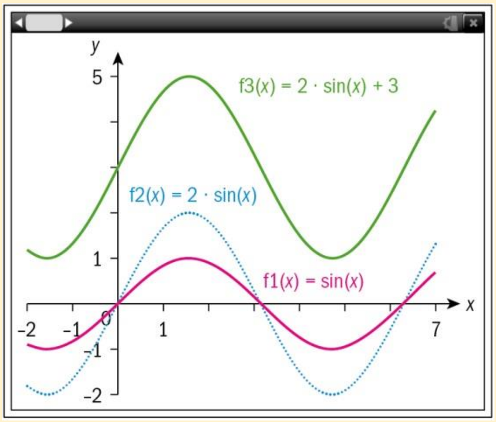


5 **Factual:** How does changing the parameter a change the graph?

Answer: Changing the value of a, changes the amplitude of the function

6 The amplitude of $y = a \sin x$ is $|a|$.

7 To map the graph of $y = \sin x$ onto the graph of $y = 2\sin x + 3$ you would first stretch vertically by a scale factor of 2, and then translate 3 in the positive vertical direction.

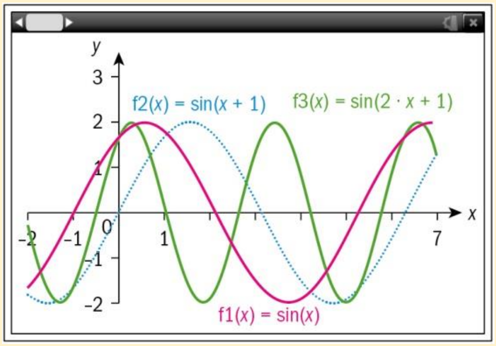


8 **Factual:** How does changing the value of *d* affect the graph?

Answer: Changing the value of *d* shifts the graph vertically.

9 To map the graph of $y = \sin x$ onto the graph of $y = \sin x - c$ you would translate *c* in the positive horizontal direction.

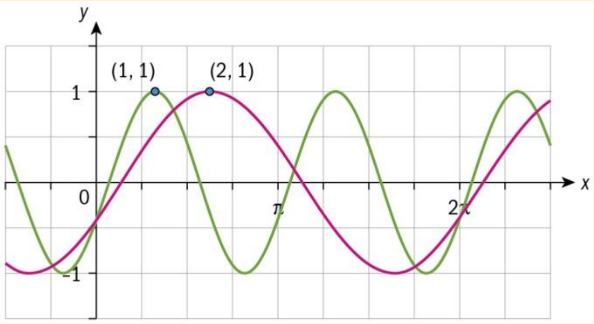
10 a



b A stretch parallel to the *x*-axis scale factor 2 is produced by the transformation function $f(2x)$ and so produces the curve $y = \cos(2x)$.

A horizontal shift of 1 is produced by the transformation function $f(x-1)$ and so if this acts on $y = \cos(2x)$ the result is $y = \cos(2(x-1))$.

11 a



b A horizontal translation of 2 is produced by the transformation function $f(x-2)$ so gives $y = \cos(x-2)$

A stretch of scale factor 2 is produced by the transformation function $f(2x)$ which when acting on $y=\cos(x-2)$ gives $y=\cos(2x-2)$

c The curve in 10b can be expanded to give $y=\cos(2x-2)$. The translations are different because a translation of 2 to the right followed by a stretch scale factor $\frac{1}{2}$ will be equivalent to a translation of 1 done after the stretch.

12 Factual: How does changing the value of the parameter *c* affect the graph?

Answer: Changing the value of *c* shifts the graph horizontally.

Conceptual: What is the difference of having values inside the sine function and outside?

Answer: Inside the sine function, everything is reversed, including the order of operations.

13 Conceptual: How does changing the parameters generally alter a trigonometric graph?

Answer: Changing the values of *c* and *d* shifts the graph horizontally and vertically respectively.

Developing inquiry skills

There are several periodic functions in the tidal graph. There is the day to day rising and falling of the tide, the height of the high tide, on two separate cycles, similarly for the height of the low tide. The model shown in red seems to model the period fairly accurately, but the amplitude is more variable.

The model is certainly a sinusoidal function and to obtain its equation we can identity its location, period and amplitude.

Question 7 in exercise 8C developed the ideas behind the daily change in the motion of the tides. Students could now investigate the yearly change in high and low tides as indicated on the diagram at the beginning of the chapter. They can do this by using date from a local port. Having found a model for the high tides, one question that they might ask is can they find the model for the low tides from this or will they need to begin the process with new data.

8.3 Completing our number system

Investigation 4

Conceptual understanding:

Complex numbers complete the number systems by allowing unsolvable quadratic equations to be solves with complex solutions occurring in conjugate pairs.

1 a $x=\dfrac{-2\pm\sqrt{2^2-4\times2\times(-1)}}{2\times2}=\dfrac{-2\pm\sqrt{12}}{4}=-\dfrac{1}{2}\pm\dfrac{1}{2}\sqrt{3}$

b $x=\dfrac{5\pm\sqrt{5^2-4\times1\times3}}{2}=\dfrac{5\pm\sqrt{13}}{2}$

2 a $x=\dfrac{8\pm\sqrt{8^2-4\times1\times16}}{2}=\dfrac{8\pm\sqrt{0}}{2}=4$

b $x = \dfrac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 5}}{2} = \dfrac{-2 \pm \sqrt{-16}}{2}$

3 $(2\text{i})^2 = -4$, $(-\text{i})^2 = -1$, $(-2\text{i})^2 = -4$, $(5\text{i})^2 = -25$, $(-5\text{i})^2 = -25$

4 $x^2 = -4 \Rightarrow x = 2\text{i}$ or $x = -2\text{i}$, $x^2 = -9 \Rightarrow x = 3\text{i}$ or $x = -3\text{i}$,
$x^2 = -2 \Rightarrow x = \sqrt{2}\text{i}$ or $x = -\sqrt{2}\text{i}$, $x^2 = -18 \Rightarrow x = 3\sqrt{2}\text{i}$ or $x = -3\sqrt{2}\text{i}$

5 $x^2 + 2x + 5 = 0 \Rightarrow x^2 + 2x = -5 \Rightarrow x^2 + 2x + 1 = -4 \Rightarrow (x+1)^2 = -4$
$\Rightarrow$ either $x + 1 = 2\text{i} \Rightarrow x = -1 + 2\text{i}$ or $x + 1 = -2\text{i} \Rightarrow x = -1 - 2\text{i}$

Factual: When does a quadratic equation have complex roots?

Answer: A quadratic equation has complex roots whenever the discriminant is less than 0.

6 $x^2 - 6x + 13 = 0 \Rightarrow (x-3)^2 = -4 \Rightarrow x = 3 \pm 2\text{i}$

$x^2 + 2x + 26 = 0 \Rightarrow (x+1)^2 = -25 \Rightarrow x = -1 \pm 5\text{i}$

$x^2 - 4x + 6 = 0 \Rightarrow (x-2)^2 = -2 \Rightarrow x = 2 \pm \sqrt{2}\text{i}$

$x^2 - 10x + 26 = 0 \Rightarrow (x-5)^2 = -1 \Rightarrow x = 5 \pm \text{i}$

7 We see that all the pairs of solutions occur in conjugate pairs. This is more apparent when using the quadratic formula.

8 **Factual:** Given one complex solution of a quadratic equation, how could you find the other?

Answer: The two complex solutions to a quadratic equation will have the same real and imaginary parts but with d different sign between them (conjugate pair).

9 **Conceptual:** Why do we need complex numbers?

Answer: Complex numbers complete the number systems by allowing unsolvable quadratic equations to be solved with complex solutions occurring in conjugate pairs.

TOK

You might want to consider the effect of graphing a periodic function and a horizontal line and searching for intercepts.

Does an infinite number of solutions lead to an answer? Can you reason what an infinite solution looks like and is this in the realms of imagination?

Investigation 5

Conceptual understandings:

We use the collection of like terms to add and subtract complex numbers.

Simplifying surds will be the same process as i behaves like a square root (of −1).

1 $2 + a - 3 + 3b + 4a = -1 + 5a + 3b$
$3 + 2\text{i} + 4 + 5\text{i} = 7 + 7\text{i}$
$a + b = 5 + 2\text{i}$, $b - c = 7 - 3\text{i}$, $2a = 4 + 6\text{i}$

$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)\text{i}$, $z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2)\text{i}$,

$pz_1 + qz_2 = (px_1 + qx_2) + (py_1 + qy_2)\text{i}$

2 **Conceptual:** What rules of algebra do we use to be able to add or subtract complex numbers?

Answer: We use the collection of like terms to add and subtract complex numbers.

3 $\left(2-3\sqrt{2}\right)\left(1+2\sqrt{2}\right) = 2 + 4\sqrt{2} - 3\sqrt{2} - 6\left(\sqrt{2}\right)^2 = 2 + \sqrt{2} - 12 = -10 + \sqrt{2}$

$(2-\text{i})(3+2\text{i}) = 6 + \text{i} - 2\text{i}^2 = 6 + \text{i} + 2 = 8 + \text{i}$

$ab = 9 + 7\text{i}$, $bc = -10 + 10\text{i}$, $a^2 = -5 + 12\text{i}$

4 **a** $\dfrac{2-3\sqrt{2}}{1+2\sqrt{2}} = \dfrac{\left(2-3\sqrt{2}\right)\left(1-2\sqrt{2}\right)}{\left(1+2\sqrt{2}\right)\left(1-2\sqrt{2}\right)} = \dfrac{14-7\sqrt{2}}{1-8} = -2 + \sqrt{2}$

b $\dfrac{2-\text{i}}{3+2\text{i}} = \dfrac{(2-\text{i})(3-2\text{i})}{(3+2\text{i})(3-2\text{i})} = \dfrac{4-7\text{i}}{13}$ or $\dfrac{4}{13} - \dfrac{7}{13}\text{i}$

$\dfrac{a}{b} = \dfrac{3+11\text{i}}{10}$, $\dfrac{b}{c} = \dfrac{-7-\text{i}}{10}$, $\dfrac{ac}{b^2} = \dfrac{-14-8\text{i}}{8-6\text{i}} = \dfrac{-16-37\text{i}}{25}$

5 **Conceptual:** How do we simplify and manipulate complex numbers?

Answer: Simplifying surds will be the same process as i behaves like a square root (of −1).

6 Answers will vary.

8.4 A geometrical interpretation of complex numbers

TOK

Links of Argand plane to Cartesian plane.

Ethics: while Argand (1806) is generally credited with the discovery, the Argand diagram was actually described by Wessel before Argand. Caspar Wessel was a Danish-Norwegian mathematician and map maker. In 1799, seven years before Argand, Wessel described the geometrical interpretation of complex numbers as points in the complex plane.

TOK

Research for a definition of imagination and imaginary. Ask students to respond to these questions:

- What are real numbers?
- How can a number be imaginary?

Investigation 6

Conceptual understanding:

Multiplying two complex numbers together results in geometrically stretching the first complex number z_1 by a factor equal to the magnitude of the second complex number z_2 and then rotating the stretched z_1 counter –clockwise by an angle equal to the argument of z_2 to arrive at the product.

Complex numbers can be represented geometrically using vectors to help add or subtract complex numbers.

1 The points represented by O, z_1, $z_1 + z_2$ and z_2 form a parallelogram. This will be true whatever values are chosen for z_1 and z_2.

2 The points represented by O, $z_1 - z_2$, z_1 and z_2 also form a parallelogram. This will be true whatever values are chosen for z_1 and z_2.

3 **Conceptual:** How can the addition and subtraction of complex numbers compare to the addition and subtraction of vectors?

Answer: If **a** and **b** are the position vectors of A and B, then $\mathbf{a}+\mathbf{b}$ is the position vector of the point P where OAPB forms a parallelogram. Also $\mathbf{a}-\mathbf{b}$ is the position vector of the point Q where OQAB forms a parallelogram.

4 Multiplying any number by i will have the effect of rotating the number by 90° anticlockwise about O. Similarly, multiplying any number by 2i will have the effect of rotating the number by 90° anticlockwise about O and moving twice as far from O (enlargement from O, SF2).

5 $\left|e^{i\theta}\right| = \left|\cos\theta + i\sin\theta\right| = \cos^2\theta + \sin^2\theta = 1$

6 **a** $|i| = 1$ $\arg(i) = \frac{\pi}{2}$ hence i can be written as $e^{\frac{\pi}{2}i}$

b $i \times re^{i\theta} = e^{i\frac{\pi}{2}} \times re^{i\theta} = re^{i\left(\frac{\pi}{2}+\theta\right)}$ This confirms the conjecture from question 4 as the argument has increased by $\frac{\pi}{2}$.

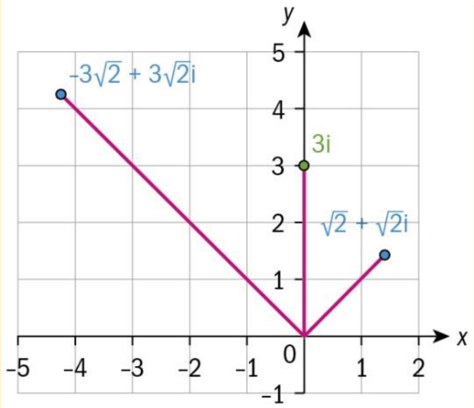
7 **a** If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ then $z_1z_2 = r_1r_2e^{i\theta_1}e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$, so the modulus is multiplied and the argument is added.

b $\frac{z_1}{z_2} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$, so the moduli are divided and the arguments subtracted.

8 Multiplying by z_1 will result in the number being rotated by θ, anticlockwise about O and stretching from O by a scale factor of r.

9 **a** $3\sqrt{2}i - 3\sqrt{2}$ or $-3\sqrt{2} + 3\sqrt{2}i$

b



c $\left|\sqrt{2}+\mathrm{i}\sqrt{2}\right|=2$ and $\arg\left(\sqrt{2}+\mathrm{i}\sqrt{2}\right)=\frac{\pi}{4}$ hence $\sqrt{2}+\sqrt{2}\mathrm{i}=2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$

Factual: What is the exponential form of a complex number?

Answer: The exponential form of a complex umber is $r\mathrm{e}^{\mathrm{i}\theta}$ where r is the modulus and q is the argument.

Conceptual: What will be the geometrical effect of multiplying z_1 by z_2 when $|z_2| = r$ and $\arg z_2 = \theta_2$?

Answer: Multiplying by z_2 will result in the number being rotated by θ_2, anticlockwise about O and stretching from O by a scale factor of r_2, if $|z_2| = r_2$ and $\arg z_2 = \theta_2$.

8.5 Using complex numbers to understand periodic models

Investigation 7

Conceptual understanding:

Understanding the geometrical behaviour of a power of a complex number can help with more complex calculations.

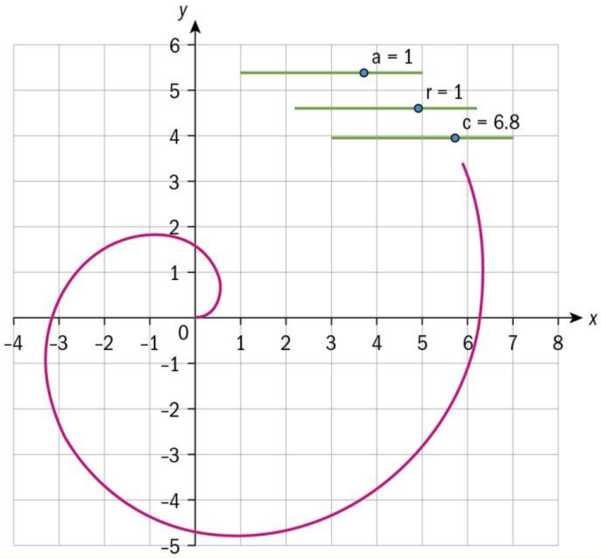
A complex number in which both the argument and the modulus are written as functions can create curves on the Argand diagram, including spirals.

The aim of this investigation is to introduce the idea of a complex number representing a function, which will be explored further in the use of phase portraits in chapter 12.

Using suitable software, set up sliders for r, a and c.

c represents the endpoint for t and should initially go from 0 to 10. By changing this, the curve is produced as t increase.

The curve should then be entered as Curve(r*t*cos(a*t), r*t*sin(a*t), t, 0, c)



Conceptual: How can complex numbers be used to create spiral curves?

Answer: A complex number in which both the argument and the modulus are written as functions can create curves on the Argand diagram, including spirals.

Investigation 8

Conceptual understandings:

The graph of the imaginary part of a complex number corresponds to a sine function and changing the modulus of the complex number alters the amplitude of the function.

The sum of two sine functions with the same frequency and amplitudes r_1 and r_2 and phase shifts a_1 and a_2 results in a sine function with a modulus and argument transformed from the original sine functions.

1 $f(\theta) = \mathrm{Im}(r_1 \operatorname{cis}\theta) = \mathrm{Im}(r_1\cos\theta + r_1 \mathrm{i}\sin\theta) = r_1\sin\theta$

2 **Conceptual:** What effect will changing the modulus of z (i.e. r_1) have on the function?

Answer: The graph of the imaginary part of a complex number corresponds to a sine function and changing the modulus of the complex number alters the amplitude of the function.

3 The function $f(\theta) = \mathrm{Im}\left(r_1 \operatorname{cis}\left(\theta + \frac{\pi}{6}\right)\right)$ is a sine function with amplitude r_1 and a phase difference of $\frac{\pi}{6}$.

4 $f(x) = a\sin(x+c) = \mathrm{Im}(a\cos(x+c) + a\mathrm{i}\sin(x+c)) = \mathrm{Im}\left(a\mathrm{e}^{\mathrm{i}(x+c)}\right) = \mathrm{Im}\left(a\mathrm{e}^{x\mathrm{i}}\mathrm{e}^{c\mathrm{i}}\right)$

5 **a** $f(x) = \sin(x) = \mathrm{Im}\left(\mathrm{e}^{x\mathrm{i}}\right)$ and $g(x) = 2\sin\left(x + \frac{\pi}{2}\right) = \mathrm{Im}\left(2\mathrm{e}^{x\mathrm{i}}\mathrm{e}^{\frac{\mathrm{i}\pi}{2}}\right)$

b So $h(x) = f(x) + g(x) = \mathrm{Im}\left(\mathrm{e}^{x\mathrm{i}}\right) + \mathrm{Im}\left(2\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\,\mathrm{e}^{x\mathrm{i}}\right) = \mathrm{Im}\left(\mathrm{e}^{x\mathrm{i}} + 2\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\mathrm{e}^{x\mathrm{i}}\right) = \mathrm{Im}\left(\mathrm{e}^{\mathrm{i}x}\left(1 + 2\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right)\right)$

c $1+2e^{i\frac{\pi}{2}}=2.24e^{1.11i}$

d $h(x)=\text{Im}\left(e^{ix}\times 2.24e^{1.11i}\right)=\text{Im}\left(2.24e^{i(x+1.11)}\right)=2.24\sin(x+1.11i)$

e Both equations will give the same curve.

f 2.24

6 If $f(x)=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$ and $g(x)=2\sin\left(x+\frac{2\pi}{3}\right)$ then

$$f(x)+g(x)=\text{Im}\left(\sqrt{2}\,e^{ix}e^{i\frac{\pi}{4}}+2e^{ix}e^{i\frac{2\pi}{3}}\right)$$

$$=\text{Im}\left(e^{ix}\left(\sqrt{2}e^{i\frac{\pi}{4}}+2e^{i\frac{2\pi}{3}}\right)\right)=\text{Im}\left(e^{ix}\left(2.73e^{1.57i}\right)\right)$$

$$=\text{Im}\left(2.73e^{(x+1.57)i}\right)=2.73\sin(x+1.57)$$

7 **Conceptual:** What can you deduce about the sum of two sine functions with the same frequency and amplitudes r_1 and r_2 and phase shifts α_1 and α_2?

Answer: The sum of two sine functions with the same frequency and amplitudes r_1 and r_2 and phase shifts α_1 and α_2 results in a sine function with a modulus and argument transformed from the original sine functions.

8 $r_1\sin(ax+\alpha_1)+r_2\sin(ax+\alpha_2)=\text{Im}\left(r_1e^{ia(x+\alpha_1)}+r_2e^{ia(x+\alpha_2)}\right)=\text{Im}\left(e^{iax}\left(r_1e^{i\alpha_1}+r_2e^{i\alpha_2}\right)\right)$

$=\text{Im}\left(e^{iax}\left(re^{i\alpha}\right)\right)=r\sin(ax+\alpha)$, where $r=\left|r_1e^{i\alpha_1}+r_2e^{i\alpha_2}\right|$ $\alpha=\arg\left(r_1e^{i\alpha_1}+r_2e^{i\alpha_2}\right)$.

TOK

Ask for definitions of accuracy and simplicity.

Consider explaining something where shared knowledge becomes personal knowledge, and language must be simplified to assist understanding.

- Does simplifying the language lose accuracy?
- Why do we use radians instead of degrees?

Making a Mandelbrot!

Approaches to Learning: Critical thinking, Communication,

Exploration Criteria: Mathematical communication (B), Personal engagement (C), Use of mathematics (E)

IB Topic: Complex numbers

A lot of students choose to look at fractals in their explorations, and so Mandelbrot and Julia sets are popular (together with the Koch snowflake and the Sierpinski Triangle). The Mandelbrot set is an intriguing construction and an interesting avenue to explore but is not always well executed if students do not take the time to understand its construction and then, importantly, to think of ways they can extend their problem.

The mathematics is relatively easy to understand. This activity is designed to help students take the first steps on this intriguing discovery.

Fractals

To introduce this topic, you could ask:

- Have you heard of fractals?
- What do you know about them?

Exploring an iterative equation

Students will now consider iterative equations to increase their understanding of how a particular fractal is created.

$z_2 = 0.5$
$z_3 = 0.75$
$z_4 = 1.0625$
$z_5 = 1.6289\ldots$
$z_6 = 3.15333\ldots$
$z_7 = 10.44352\ldots$

The values found in this calculation are getting larger and larger – towards infinity.

A different iterative equation

$z_2 = -0.5$
$z_3 = -0.25$
$z_4 = -0.4375$
$z_5 = -0.30859\ldots$
$z_6 = -0.40476\ldots$
$z_7 = -0.33616\ldots$

The values found in this calculation stay within a boundary, converging.

The Mandelbrot set

You could ask:

- Can you find another integer value of c that would mean that the process would zoom off to infinity?

e.g. $C=1$

All values

Can you find another integer value c where the process wouldn't zoom off to infinity?

e.g. $C = -1.3, -1.1$

$z_2 = 1 + \mathrm{i}$

$z_3 = 1 + 3\mathrm{i}$

$z_4 = -7 + 7\mathrm{i}$

$z_5 = 1 - 97i$

$z_6 = -9407 - 193i$

$z_7 = 88\,454\,401 - 3\,631\,103i$

Clearly this zooms off to infinity.

a $c = 0.2 - 0.7i$

a	+	b	i
0	+	0	i
0.2	+	-0.7	i
-0.25	+	-0.98	i
-0.6979	+	-0.21	i
0.6429644	+	-0.406882	i
0.4478503	+	-1.2232213	i
-1.0957005	+	-1.79564	i

This is heading off to infinity (slowly), so is not in the Mandelbrot set.

b $c = -0.25 + 0.5i$

a	+	b	i
0	+	0	i
-0.25	+	0.5	i
-0.4375	+	0.25	i
-0.1210938	+	0.28125	i
-0.3144379	+	0.4318848	i
-0.3376533	+	0.2283982	i
-0.188156	+	0.3457612	i
-0.3341482	+	0.3698859	i
-0.2751606	+	0.2528066	i
-0.2381978	+	0.3608752	i
-0.3234927	+	0.3280806	i
-0.2529894	+	0.2877366	i
-0.2687887	+	0.3544114	i
-0.30336	+	0.3094764	i
-0.2537484	+	0.3122344	i
-0.2831021	+	0.3415421	i
-0.2865042	+	0.3066174	i
-0.2619296	+	0.3243057	i
-0.286567	+	0.3301095	i
-0.2768516	+	0.310803	i
-0.2699517	+	0.3279074	i
-0.2846493	+	0.3229617	i
-0.273279	+	0.3161383	i

This is not heading off to infinity, so is in the Mandelbrot set.

If $c = i$:

a	+	b	i
0	+	0	i
0	+	1	i
-1	+	1	i
0	+	-1	i
-1	+	1	i
0	+	-1	i
-1	+	1	i
0	+	-1	i
-1	+	1	i
0	+	-1	i
-1	+	1	i
0	+	-1	i

This is not going to infinity, so is in the Mandelbrot set.

Students could find an efficient way of trying some of their own values on a calculator, or start thinking about a spreadsheet to do the calculations. There will also hopefully be a comment regarding the fact that it is easier to spot the values heading off to infinity (or not) with some values rather than others.

With a class it can be good to have a set of axes for an Argand diagram on the board (say from −2 to 2 on the real axis and −2 to 2 on the imaginary axes). Students can then come up to the board when they have calculated a point and colour it according to whether it is in the Mandelbrot set or not – slowly (very slowly!) the set will take shape!

Here is an example of a programme on Geogebra: https://www.geogebra.org/m/vK8zhJKM.

Extensions

The Mandelbrot and the related Julia set is a popular topic for mathematics explorations. It is important therefore that students do more than just explain how it is formed.

You could ask:

- What other questions could you ask?
- What other approaches could you pursue to make your own exploration?

A few suggestions are given in the task in the Students Book.

Some of these extensions may take students well beyond mathematics they are comfortable with, and therefore they may find it difficult to demonstrate understanding. You may need to advise them carefully on this if their ideas are too ambitious.

9 Modelling with matrices: storing and analysing data

Essential understandings

Number and algebra allow us to represent patterns, show equivalencies and make generalizations which enable us to model real-world situations. Algebra is an abstraction of numerical concepts and employs variables to solve mathematical problems.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Matrices allow us to organize data so that they can be manipulated and relationships can be determined.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
The product of two matrices exists when the number of columns in the first matrix corresponds to the number of rows in the second matrix.	Investigation 1
Matrix addition follows commutative and associative properties but unlike multiplication of real numbers, matrix multiplication follows associative property and not the commutative property.	Investigation 2
Independent events represent unrelated, separate events where the probability of one events does not affect the outcome of the other.	Investigation 8
Probabilities for Markov chains can be represented by a transition matrix and the probabilities for the various states of the system after n transitions can be found from the entries of the transition matrix raised to the power of n.	Investigation 9
For a regular Markov chain with transition matrix **T** as n tends to infinity the matrix $\mathbf{T}^n$ tends to a matrix in which all the columns have equal entries.	Investigation 10
The powers of the transition matrix may be used to make predictions about the state of a population in the short and long term.	Investigation 11
The steady state vector $\boldsymbol{u} = \binom{u_1}{u_2}$ can be found either by finding the steady state transition matrix through considering high powers of $\mathbf{T}^n$ and then multiplying this by the initial distribution vector, or by solving the equations $\mathbf{Tu} = \mathbf{u}$, and $u_1 + u_2 = p$ where p is the total initial population.	Investigation 12

Syllabus sections covered in this chapter:

- AHL1.14
- AHL1.15
- AHL3.9

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

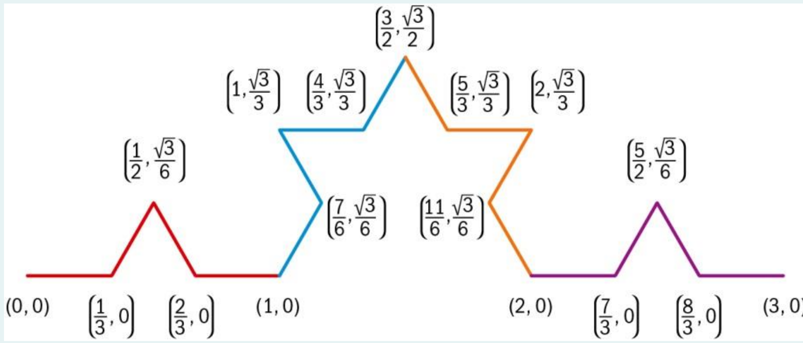
Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 373	Page 394: Example 7 Page 417, Example 13	Page 386: Example 4 Page387: Example: 5 Page 404, Example: 10 Page 409, Example: 11 Page 417, Example: 13	Pages 376, 383, 389, 400, 405, 412, 420

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 420	Page 422	N/A

Developing inquiry skills

The values of the coordinates for the opening question are shown in the diagram below.



Where else do fractals appear in the real world?

Can we use fractal formulae to model real-life situations? How might these be useful?

Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Answer: Fractals (or more specifically self-similarity, or unfolding similarity) is present in many places in the natural world. Fractals were defined generally by Mandelbrot as "A fractal is a shape made of parts similar to the whole in some way."

It was first considered by mathematicians when trying to calculate the length of a border of a country and it was realized the answer obtained dependent on the length of the measuring tool used.

Fractals are also considered as appearing in nature where self-similarity is present over an extended range. Some examples of these include: ferns, the structure of the lung or blood vessels, crystals, DNA, etc.

The consideration of fractal dimensions (though not in the syllabus) could be discussed and linked with work on logarithms.

9.1 Introduction to matrices and matrix operations

Section 9.1 is an introduction to the idea of a matrix as an array of numbers, including addition of matrices and multiplication by a scalar. These operations are linked to practical examples.

TOK

"There is no branch of mathematics, however abstract, which may not someday be applied to phenomena of the real world." – Nikolai Lobatchevsky

Where does the power of mathematics come from? Is it from its ability to communicate as a language, from the axiomatic proofs or from its abstract nature?

A great chance for academic students to experience the complexities of mathematics. The nature of mathematics. reveals hidden patterns that help us understand the world around us.

Research an abstraction in mathematics, axioms and mathematical language.

Matrices treat abstract objects, mathematics often relies on reason but there is a place for sense perception, imagination, and even intuition.

9.2 Matrix multiplication and properties

Investigation 1

Conceptual understanding:

The product of two matrices exists when the number of columns in the first matrix corresponds to the number of rows in the second matrix.

1 **Factual:** Find the products of ***AB***, ***AC***, ***BD***, ***DE***, and ***BE***.

Answer:

$$\boldsymbol{AB}=\begin{pmatrix}4 & -14 & 22\\ -2 & -2 & 1\end{pmatrix},\ \boldsymbol{AC}=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix},\ \boldsymbol{BD}=\begin{pmatrix}-1 & -8 & -2\\ -4 & -21 & 1\end{pmatrix},$$

$$\boldsymbol{DE}=\begin{pmatrix}2\\ 2\\ -2\end{pmatrix},\ \boldsymbol{BE}=\begin{pmatrix}-8\\ -7\end{pmatrix}$$

2

Product	Dimensions of first matrix	Dimensions of second matrix	Dimensions of product
AB	2×2	2×3	2×3
AC	2×2	2×1	2×1
BD	2×3	3×3	2×3
DE	3×3	3×1	3×1
BE	2×3	3×1	2×1

3 **Factual:** Based upon the results in question **1** explain how you could determine the dimensions of the product of two matrices using the dimensions of the two matrices being multiplied?

Answer: The dimensions of the product will be the number of rows in the first matrix by the number of columns in the second matrix.

4 The products ***BA***, ***CB***, and ***DB*** cannot be determined since number of columns in the first matrix is not equal to the number of rows in the second matrix.

5 **Conceptual:** When does the product of two matrices exist?

Answer: The product of two matrices exists when the number of columns in the first matrix corresponds to the number of rows in the second matrix.

Investigation 2

Conceptual understanding:

Matrix addition follows commutative and associative properties but unlike multiplication of real numbers, matrix multiplication follows associative property and not the commutative property.

1 $\boldsymbol{AB} = \begin{pmatrix} -16 & -4 \\ 2 & 18 \end{pmatrix}$ and $\boldsymbol{BA} = \begin{pmatrix} 22 & 8 \\ -20 & -20 \end{pmatrix}$ and thus matrix multiplication is not commutative.

2 **i** $\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \begin{pmatrix} -21 & 6 \\ -3 & 18 \end{pmatrix}$

ii $(\boldsymbol{B} + \boldsymbol{C})\boldsymbol{A} = \begin{pmatrix} 30 & 15 \\ -42 & -33 \end{pmatrix}$

iii $\boldsymbol{AB} + \boldsymbol{AC} = \begin{pmatrix} -21 & 6 \\ -3 & 18 \end{pmatrix}$

iv $\boldsymbol{BA} + \boldsymbol{CA} = \begin{pmatrix} 30 & 15 \\ -42 & -33 \end{pmatrix}$

3 Matrix multiplication is distributive provided that we recognize that these products are not commutative. That is, $\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$ and $(\boldsymbol{B} + \boldsymbol{C})\boldsymbol{A} = \boldsymbol{BA} + \boldsymbol{CA}$. It is therefore incorrect to write $(\boldsymbol{B} + \boldsymbol{C})\boldsymbol{A} = \boldsymbol{AB} + \boldsymbol{CA}$.

4 $\boldsymbol{A}(\boldsymbol{BC}) = \begin{pmatrix} -2 & -3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 17 & -21 \\ -18 & 14 \end{pmatrix} = \begin{pmatrix} 20 & 0 \\ 50 & -70 \end{pmatrix}$ and $(\boldsymbol{AB})\boldsymbol{C} = \begin{pmatrix} -16 & -4 \\ 2 & 18 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 20 & 0 \\ 50 & -70 \end{pmatrix}$.

Matrix multiplication is associative.

5 **a** $\boldsymbol{AI} = \begin{pmatrix} -2 & -3 \\ 4 & 1 \end{pmatrix}$, $\boldsymbol{IA} = \begin{pmatrix} -2 & -3 \\ 4 & 1 \end{pmatrix}$

b $\boldsymbol{AI}$ and $\boldsymbol{IA}$ are both equal to $\boldsymbol{A}$.

6 **Conceptual:** How do matrix and matrix multiplication work in terms of the commutative and associative properties?

Answer: Matrix addition follows commutative and associative properties but unlike multiplication of real numbers, matrix multiplication follows associative property and not the commutative property.

Investigation 3

1 $\boldsymbol{A}^2 = \begin{pmatrix} 2 & -3 \\ 4 & 7 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 4 & 7 \end{pmatrix} = \begin{pmatrix} -8 & -27 \\ 36 & 37 \end{pmatrix} \neq \begin{pmatrix} 2^2 & (-3)^2 \\ 4^2 & 7^2 \end{pmatrix}$

2 $\boldsymbol{B}^2 = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$, $\boldsymbol{C}^2 = \begin{pmatrix} 0 & 2 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}$,

$\boldsymbol{I}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3 $\boldsymbol{A}^2 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a^2 & 0^2 \\ 0^2 & d^2 \end{pmatrix}$.

In general, this holds if $\boldsymbol{A}$ is an $n \times n$ matrix whose only non-zero elements lie along the main diagonal. For example,

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -0.5 \end{pmatrix}^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0.25 \end{pmatrix}^2$$

By the definition of matrix multiplication, element $a_{i,j}$ in the product of $\boldsymbol{A}$ with $\boldsymbol{A}$ is obtained by summing the products of row i of $\boldsymbol{A}$ with column j of $\boldsymbol{A}$. Since the only non-zero entries are elements $a_{i,i}$ (or equivalently $a_{j,j}$) then the only non-zero entries in the product would be obtained by multiplying $a_{i,i}$ by itself giving $(a_{i,i})^2$.

4 $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}^3 = \begin{pmatrix} 8 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}^4 = \begin{pmatrix} 16 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}^k = \begin{pmatrix} 2^k & 0 \\ 0 & 1 \end{pmatrix}$

5 For real numbers a, $a \neq 0$, $a^0 = 1$ (= the identity for real numbers.) Thus, a reasonable definition for $\boldsymbol{A}^0$ might be $\boldsymbol{A}^0 = \boldsymbol{I}_n$. The GDC also gives this result. When $\boldsymbol{A} = \boldsymbol{0}$ the GDC gives $\boldsymbol{A}^0 = \boldsymbol{I}_n$ which contradicts the fact that 0^0 is undefined for real numbers.

9.3 Solving systems of equations using matrices

Investigation 4

a i Let $\boldsymbol{B} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ be the multiplicative inverse of $\boldsymbol{A}$. Use the fact that $\boldsymbol{AB} = \boldsymbol{I}$ to show that

$$\left\{\begin{matrix} 8w - 6y = 1 \\ -5w + 4y = 0 \end{matrix}\right\} \text{ and } \left\{\begin{matrix} 8x - 6z = 0 \\ -5x + 4z = 1 \end{matrix}\right\}.$$

$$\boldsymbol{AB} = \boldsymbol{I}$$

$$\begin{pmatrix} 8 & -6 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 8w - 6y & 8x - 6y \\ -5w + 4y & -5x + 4z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Equating elements gives $\begin{matrix} 8w - 6y = 1 \\ -5w + 4y = 0 \end{matrix}$ and $\begin{matrix} 8x - 6z = 0 \\ -5x + 4z = 1 \end{matrix}$.

ii $\left\{\begin{matrix} 8w - 6y = 1 \\ -5w + 4y = 0 \end{matrix}\right\} \begin{matrix} \times 5 \\ \times 8 \end{matrix} \Leftrightarrow \left\{\begin{matrix} 40w - 30y = 5 \\ -40w + 32y = 0 \end{matrix}\right\} \Leftrightarrow 2y = 5 \quad \therefore y = \frac{5}{2}, w = 2$

$\left\{\begin{matrix} 8x - 6z = 0 \\ -5x + 4z = 1 \end{matrix}\right\} \begin{matrix} \times 5 \\ \times 8 \end{matrix} \Leftrightarrow \left\{\begin{matrix} 40x - 30z = 0 \\ -40x + 32z = 8 \end{matrix}\right\} \Leftrightarrow 2z = 8 \quad \therefore z = 4, x = 3$

Thus, $\boldsymbol{B} = \begin{pmatrix} 2 & 3 \\ \frac{5}{2} & 4 \end{pmatrix}$.

$\boldsymbol{AB} = \begin{pmatrix} 8 & -6 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ \frac{5}{2} & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{I}^2$ and $\boldsymbol{BA} = \begin{pmatrix} 2 & 3 \\ \frac{5}{2} & 4 \end{pmatrix}\begin{pmatrix} 8 & -6 \\ -5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{I}^2$.

b i $\boldsymbol{AB} = \boldsymbol{I}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} aw+by & ax+bz \\ cw+dy & ax+dz \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Equating elements gives $\left\{\begin{matrix} aw+by=1 \\ cw+dy=0 \end{matrix}\right\}$ and $\left\{\begin{matrix} ax+bz=0 \\ cx+dz=1 \end{matrix}\right\}$.

ii

$$\left\{\begin{matrix} aw+by=1 \\ cw+dy=0 \end{matrix}\right\}\begin{matrix} \times c \\ \times(-a) \end{matrix} \Leftrightarrow \left\{\begin{matrix} acw+bcy=c \\ -acw-ady=0 \end{matrix}\right\} \Leftrightarrow (bc-ad)y=c \quad \therefore y=\frac{c}{(bc-ad)}=\frac{-c}{(ad-bc)}$$

$$\left\{\begin{matrix} aw+by=1 \\ cw+dy=0 \end{matrix}\right\}\begin{matrix} \times d \\ \times(-b) \end{matrix} \Leftrightarrow \left\{\begin{matrix} adw+bdy=d \\ -bcw-bdy=0 \end{matrix}\right\} \Leftrightarrow (ad-bc)w=d \quad \therefore w=\frac{d}{(ad-bc)}$$

$$\left\{\begin{matrix} ax+bz=0 \\ cx+dz=1 \end{matrix}\right\}\begin{matrix} \times c \\ \times(-a) \end{matrix} \Leftrightarrow \left\{\begin{matrix} acx+bcz=0 \\ -acx-adz=-a \end{matrix}\right\} \Leftrightarrow (bc-ad)z=-a \quad \therefore z=\frac{-a}{(bc-ad)}=\frac{a}{(ad-bc)}$$

$$\left\{\begin{matrix} ax+bz=0 \\ cx+dz=1 \end{matrix}\right\}\begin{matrix} \times d \\ \times(-b) \end{matrix} \Leftrightarrow \left\{\begin{matrix} adx+bdz=0 \\ -bcx-bdz=-b \end{matrix}\right\} \Leftrightarrow (ad-bc)x=-b \quad \therefore x=\frac{-b}{(ad-bc)}$$

Thus $\boldsymbol{B} = \begin{pmatrix} \frac{d}{(ad-bc)} & \frac{-b}{(ad-bc)} \\ \frac{-c}{(ad-bc)} & \frac{a}{(ad-bc)} \end{pmatrix} = \frac{1}{(ad-bc)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$\boldsymbol{AB}$ =

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \frac{1}{(ad-bc)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{(ad-bc)}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & d \end{pmatrix}\right) = \frac{1}{(ad-bc)}\begin{pmatrix} ad-bc & cd-cd \\ -ab+ab & -bc=ad \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boldsymbol{BA} = \frac{1}{(ad-bc)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{(ad-bc)}\left(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).$$

$$= \frac{1}{(ad-bc)}\begin{pmatrix} ad-bc & bd-bd \\ -ac+ac & -bc+ad \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

TOK

Do you think that one form of symbolic representation is preferable to another?

The symbols used internationally for mathematics differ across countries and teachers.

Research and display some such as $y =$ and $f(x)$, the different ways of showing the domain, some using a comma as a decimal point, etc.

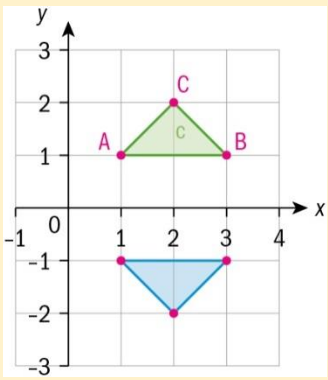
9.4 Transformations of the plane

Investigation 5

The aim of the investigation is to derive many of the standard transformation matrices. It should be noted that all of these are given in the formula books.

1 $\begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix}$

2



The transformation is a reflection in the x-axis.

3 $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

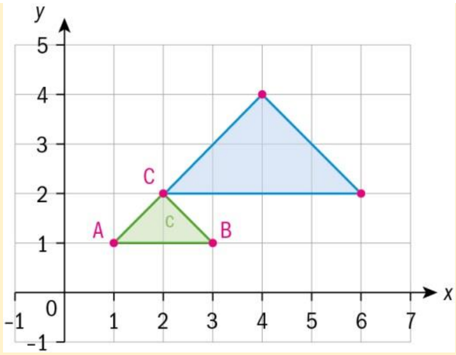
The image matrix is the same as the transformation matrix.

4 The image matrix is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This will always be the case as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix.

5 The image of (1, 0) is (2, 0) and the image of (0, 1) is (0, 2) under an enlargement scale factor 2, center (0, 0).

The matrix will therefore be $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

We can check by doing $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 2 & 4 & 2 \end{pmatrix}$ and hence the triangle is also enlarged scale factor 2.



6 Under a 90° clockwise rotation the image of (1, 0) is (0, −1) and the image of (0, 1) is (1, 0).

Hence the matrix is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

7 Using the same method the matrix is $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

8 **a** The point A is the image of (1, 0). Using right-angled trigonometry and the fact that the hypotenuse is 1 gives us the coordinates of the image of (1, 0) as $(-\cos\theta, \sin\theta)$. Similarly the image of (0, 1) is $(-\sin\theta, \cos\theta)$ so the transformation matrix is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

b A similar diagram can be drawn and the matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ obtained.

c $\begin{pmatrix} \cos 60^\circ & \sin 60^\circ \\ -\sin 60^\circ & \cos 60^\circ \end{pmatrix} = \begin{pmatrix} 0.5 & 0.866 \\ -0.866 & 0.5 \end{pmatrix}$

9 **a** Because the angle AOC is 2α and the length OA is equal to 1.

b The angle between the y-axis and the line $y = (\tan\alpha)x$ is $90 - \alpha$. This will be the same as the angle between $y = (\tan\alpha)x$ and [OB]. Hence angle BOC is $90 - \alpha - \alpha = 90 - 2\alpha$ and so angle $\mathrm{O\hat{B}C}$ is equal to 2α.

Hence the image of (0, 1) is $(\sin 2\alpha, -\cos 2\alpha)$.

c The matrix comes directly from the coordinates of the images of (1, 0) and (0, 1).

d $\tan\alpha = \sqrt{3} \Rightarrow \alpha = 60^\circ$

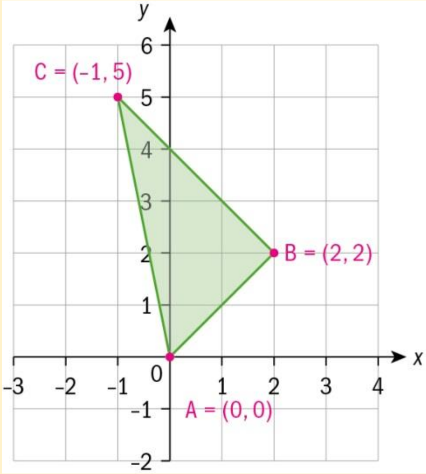
$\begin{pmatrix} -0.5 & 0.866 \\ 0.866 & 0.5 \end{pmatrix}$

10 **a** $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$

b $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

Investigation 6

1



Slope of AB = 1 and slope of BC = −1 thus since the product of the slope is −1 then $AB \perp BC$

AB = $\sqrt{2^2 + 2^2} = \sqrt{8}$

BC = $\sqrt{3^2 + (-3)^2} = \sqrt{18}$

Area of ABC = $\frac{1}{2} AB \cdot BC = \frac{1}{1}\sqrt{144}$ = 6

2 **a** $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ and thus

A(0, 0) → A′(0, 0)

B(2, 2) → B′(6, 6)

C(−1, 5) → C′(−3, 15)

b Note that A’B’C’ forms a right triangle.

A’B’ = $\sqrt{6^2 + 6^2} = \sqrt{72}$

B’C’ = $\sqrt{9^2 + (-9)^2} = \sqrt{162}$

Area of A’B’C’ = $\frac{1}{2}$A’B’ × B’C’ $\frac{1}{2}\sqrt{11664}$ = 54

Area of triangle A’B’C’ = 9 × area of triangle ABC.

Enlarging each point by a factor of k increases the lengths of each side of the triangle by a factor of k and thus increases the area by a factor of k^2.

3 **a** Use matrix multiplication to determine the coordinates of A’B’C’.

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ and thus

A(0, 0) → A′(0, 0)

B(2, 2) → B′(6, 2)

C(−1, 5) → C′(9, −7)

b A′B′C′ forms a right triangle since $A'B' \perp B'C'$ (slopes are $\frac{1}{3}$ and −3)

A′B′ = $\sqrt{2^2+6^2} = \sqrt{40}$

B′C′ = $\sqrt{3^2+(-9)^2} = \sqrt{90}$

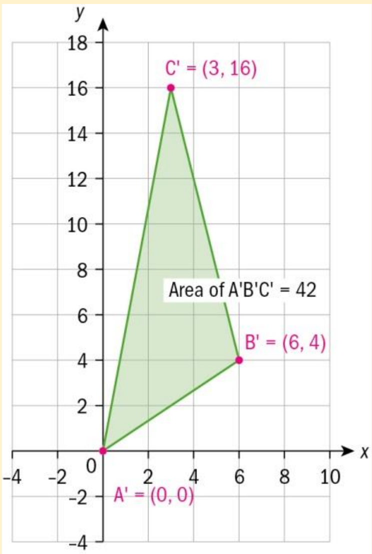
Area of A′B′C′ = $\frac{1}{2}$A′B′ × B′C′ = $\frac{1}{2}\sqrt{360}$ = 30

Area of triangle A′B′C′ = 5 × area of triangle ABC.

c det(T) = −1 − 4 = −5

Area of triangle A′B′C′ = 30 = |5| × area of triangle ABC.

4 a



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$ and thus

A(0, 0) → A′(0, 0)

B(2, 2) → B′(6, 4)

C(−1, 5) → C′(3, 16)

The product of the slopes of any two consecutive sides is not equal to −1 and thus no two sides are perpendicular.

b det(T) = 6 − (−1) =7

Area of triangle A′B′C′ = 42 × |7| × area of triangle ABC.

5 $$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and thus A(0, 0) → A′(0, 0)

B(2, 2) → B′(6, 18)

C(−1, 5) → C′(3, 9)

Slopes of A′B′ and B′C′ are both equal to 3 and thus A′B′C′ are collinear. A′B′C′ do not form a triangle and this case could reason that the triangle has 0 area. Thus,

det(T) = 6 − 6 =0

Area of triangle A′B′C′ = 0 × |0| × area of triangle ABC.

and therefore the relationship holds.

Investigation 7

This investigation introduces the idea of using affine transformations to produce self-similar objects. It combines the work done with matrices with previous work on geometric series.

1 For (0, 0)

$$\begin{pmatrix}8\\0\end{pmatrix} = \boldsymbol{A}\begin{pmatrix}0\\0\end{pmatrix} + \boldsymbol{b}$$

hence

$$\boldsymbol{b} = \begin{pmatrix}8\\0\end{pmatrix}$$

For (8,0)

$$\begin{pmatrix}12\\0\end{pmatrix} = \boldsymbol{A}\begin{pmatrix}8\\0\end{pmatrix} + \begin{pmatrix}8\\0\end{pmatrix} \Rightarrow \boldsymbol{A}\begin{pmatrix}8\\0\end{pmatrix} = \begin{pmatrix}4\\0\end{pmatrix}$$

For (0, 8)

$$\begin{pmatrix}8\\4\end{pmatrix} = \boldsymbol{A}\begin{pmatrix}0\\8\end{pmatrix} + \begin{pmatrix}8\\0\end{pmatrix} \Rightarrow \boldsymbol{A}\begin{pmatrix}0\\8\end{pmatrix} = \begin{pmatrix}0\\4\end{pmatrix}$$

Hence

$$\boldsymbol{A}\begin{pmatrix}8 & 0\\0 & 8\end{pmatrix} = \begin{pmatrix}4 & 0\\0 & 4\end{pmatrix} \Rightarrow \boldsymbol{A} = \begin{pmatrix}0.5 & 0\\0 & 0.5\end{pmatrix}$$

2 Any point can be chosen for the verification, it does not matter if the choice is of a point on the boundary rather than in the interior. A common choice is likely to be (10, 2)

$$\begin{pmatrix}0.5 & 0\\0 & 0.5\end{pmatrix}\begin{pmatrix}10\\2\end{pmatrix} + \begin{pmatrix}8\\0\end{pmatrix} = \begin{pmatrix}13\\1\end{pmatrix}.$$

That this gives the correct position for the image is easily verified from the diagram. This indicates that the transformation is valid between any pair of adjacent squares.

3 This question is to lead the students towards an iterative approach. (6, 6) is in S_0 so the image in S_1 can be found, then the image in S_2 and then S_3. Using the transformation found in question **1** the points are: (11, 3), (13.5, 1.5), (14.75, 0.75)

4 **a** $$\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}0.5 & 0\\0 & 0.5\end{pmatrix}\begin{pmatrix}x_0\\y_0\end{pmatrix} + \begin{pmatrix}8\\0\end{pmatrix} = \begin{pmatrix}0.5x_0 + 8\\0.5y_0\end{pmatrix}$$

4 b $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} 8 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} 0.5x_0 + 8 \\ 0.5y_0 \end{pmatrix} + \begin{pmatrix} 8 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}x_0 + 4 + 8 \\ \frac{1}{4}y_0 \end{pmatrix}$

c $\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} 8 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} \frac{1}{4}x_0 + 4 + 8 \\ \frac{1}{4}y_0 \end{pmatrix} + \begin{pmatrix} 8 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{8}x_0 + 2 + 4 + 8 \\ \frac{1}{8}y_0 \end{pmatrix}$

d The coefficients of x_0 and y_0 form a geometric sequence with a common ration of $\frac{1}{2}$. The coefficient of x_n and y_n is $\frac{1}{2^n}$ so $y_n = \frac{1}{2^n}y_0$.

In addition

$$x_n = \frac{1}{2^n}x_0 + 8 + 4 + \ldots + \frac{8}{2^{n-1}}.$$

Using the formula for the sum of a geometric series this can be written as

$$x_n = \frac{1}{2^n}x_0 + \frac{8\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} = \frac{1}{2^n}x_0 + 16\left(1 - \frac{1}{2^n}\right)$$

5 The image of (6, 2) will be

$$\left(\frac{6}{2^4} + 16\left(1 - \frac{1}{2^4}\right), \frac{2}{2^4}\right) = (15.375, 0.125)$$

It would be good practice for the students to also verify the formula with at least one of the points worked out in the previous questions.

Developing inquiry skills

1 Describe the series of transformations that are necessary to map stage 1 to the red image in stage 2. Write your answer in the form of ***AX*** + ***b*** where ***A*** is a 2 × 2 matrix, ***b*** is a 2× 1 column vector, and ***X*** is a 2 × 1 column vector representing the points of the vertices and endpoints of stage 1. Name this transformation T_1. By applying T_1 to each vertex and endpoint in stage 1 verify that the image points are those of the red piece in stage 2.

Answer:

T_1: Expand the points in stage 1 by a factor of $\frac{1}{3}$ to obtain the **red piece**.

$$T_1 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$T_1: \begin{pmatrix} 0 & 1 & \frac{3}{2} & 2 & 3 \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & 1 \\ 0 & 0 & \frac{\sqrt{3}}{6} & 0 & 0 \end{pmatrix}$$

and the image points agree with those we found geometrically.

2 Repeat the steps in question**1** by determining transformations of the form ***AX*** + ***b*** that maps the vertices in stage 1 onto the blue, orange, and purple pieces of stage 2. Name these transformations T_2, T_3 and T_4 respectively. By applying these transformations determine the coordinates of each vertex and endpoint in each stage.

Answer:

T_2: Expand the points in stage 1 by a factor of $\frac{1}{3}$ then rotate the points about the origin anti-clockwise by 60° then translate the points by the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to obtain the **blue piece.**

$$T_2 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & -\frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$T_2: \begin{pmatrix} 0 & 1 & \frac{3}{2} & 2 & 3 \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{7}{6} & 1 & \frac{4}{3} & \frac{3}{2} \\ 0 & \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

and the image points agree with those we found geometrically.

T_3: Expand the points in stage 1 by a factor of $\frac{1}{3}$ then rotate the points about the origin clockwise by 60° (anti-clockwise by 300°) then translate the points by the vector $\begin{pmatrix} \frac{3}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$ to obtain the **orange piece**.

$$T_3 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{1} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{6} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{3}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$T_3: \begin{pmatrix} 0 & 1 & \frac{3}{2} & 2 & 3 \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \end{pmatrix} \to \begin{pmatrix} \frac{3}{2} & \frac{5}{3} & 2 & \frac{11}{6} & 2 \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{6} & 0 \end{pmatrix}$$

and the image points agree with those we found geometrically.

T_4: Expand the points in stage 1 by a factor of $\frac{1}{3}$ then translate the points by the vector $\begin{pmatrix}2\\0\end{pmatrix}$ to obtain the **purple piece**.

$$T_4 = \begin{pmatrix}\frac{1}{3} & 0\\ 0 & \frac{1}{3}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}2\\0\end{pmatrix}$$

$$T_4: \begin{pmatrix}0 & 1 & \frac{3}{2} & 2 & 3\\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0\end{pmatrix} \to \begin{pmatrix}2 & \frac{7}{3} & \frac{5}{2} & \frac{8}{3} & 3\\ 0 & 0 & \frac{\sqrt{3}}{6} & 0 & 0\end{pmatrix}$$

$$T_4 = \begin{bmatrix}\frac{1}{3} & 0\\ 0 & \frac{1}{3}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}2\\0\end{bmatrix}$$

$$T_4: \begin{bmatrix}0 & 1 & \frac{3}{2} & 2 & 3\\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0\end{bmatrix} \to \begin{bmatrix}2 & \frac{7}{3} & \frac{5}{2} & \frac{8}{3} & 3\\ 0 & 0 & \frac{\sqrt{3}}{6} & 0 & 0\end{bmatrix}$$

and the image points agree with those we found geometrically.

9.5 Representing systems

Investigation 8

Conceptual understanding:

Independent events represent unrelated, separate events where the probability of one events does not affect the outcome of the other.

Sequence B:

$P(\text{Wet tomorrow} \mid \text{Wet today}) = \frac{9}{13}$ and $P(\text{Dry tomorrow} \mid \text{Wet today}) = \frac{4}{13}$

$P(\text{Wet tomorrow} \mid \text{Dry today}) = \frac{4}{7}$ and $P(\text{Dry tomorrow} \mid \text{Dry today}) = \frac{3}{7}$

1 When tossing a coin the results of each toss should be independent of the result of the previous toss. Hence P(wet tomorrow|dry today) should be approximately equal to P(wet tomorrow| wet today) and both of these should be approximately equal to the P(wet tomorrow) which would be 0.5 in the case of the coin toss. For B these probabilities are $\frac{5}{12}$ and $\frac{3}{7}$, and so there is some indication this sequence is the result of the coin toss.

2 In this small sample, one cannot be certain because the run could just be due to random behaviour. Sequence A looks like it may be from the computer simulation since the conditional probabilities are – arguably – consistent with the probabilities programmed into the simulation.

3 **Conceptual:** How do you describe independent events?

Answer: Independent events represent unrelated, separate events where the probability of one events does not affect the outcome of the other.

TOK

How do "believing that" and "believing in" differ?

How does belief differ from knowledge?

You might want to read from the Gifford Lectures.

"Believing that" means that you think something is real. Believing can be a personal opinion and it means that you think that it is more concrete.

"Believing in" means that what you believe in is more personal and less concrete. Believing in is more hopeful and gives a sense of fulfilment.

Have students share several examples of each with a partner.

How does the teacher's shared knowledge fit into this? Is it something that students "believe that", "believe in" or trust?

How does trust play a role?

Investigation 9

Conceptual understanding:

Probabilities for Markov chains can be represented by a transition matrix and the probabilities for the various states of the system after n transitions can be found from the entries of the transition matrix raised to the power of n.

1 0.45

2 0.55

3 $T^2 = \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{pmatrix} \times \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.56 & 0.55 \\ 0.44 & 0.45 \end{pmatrix}$

4 The entries in the matrix are the same as the probabilities found at the end of each branch of the tree drawn in **1** and **2**. These entries show each probability of each future state after two days given each current state.

5 **Factual:** What is more efficient to find the probabilities in T^2 – matrix multiplication or drawing the two tree diagrams?

Answer: Matrix multiplication.

6 **Factual:** How do you interpret the values in the matrix T^3?

Answer: The entries show each probability of each future state after three days given each current state.

7 **Factual:** How can we interpret our results when multiplying by the transition matrix?

Answer: Each multiplication by the transition matrix adds another time period.

8 **Factual:** What does n represent in T^n?

Answer: The power of the transition matrix represents the number of time periods.

9 **Conceptual:** How can the probabilities of Markov chains be represented?

Answer: Probabilities for Markov chains can be represented by a transition matrix and the probabilities for the various states of the system after n transitions can be found from the entries of the transition matrix raised to the power of n.

9.6 Representing steady state systems

TOK

If we can find solutions of higher dimensions, can we reason that these spaces exist beyond our sense perception?

We are looking towards using sense, perception and reason.

Given the many applications of matrices in this course, consider the fact that mathematicians marvel at some of the deep connections between disparate parts of their subject.

Is this evidence for a simple underlying mathematical reality?

Investigation 10

Conceptual understanding:

For a regular Markov chain with transition matrix $\boldsymbol{T}$ as n tends to infinity the matrix $\boldsymbol{T}^n$ tends to a matrix in which all the columns have equal entries.

1 Using the GDC to find powers of each transition matrix $\boldsymbol{T}$.

2 **a** has a converging pattern

b has a converging pattern

c has an alternating pattern

d converges

e converges

3 A regular transition matrix is one for which there is no deterministic pattern in the sequence of matrices. A periodic transition matrix is one where the sequence of matrices has a repeating pattern.

4 **a** regular

b regular

c periodic

d absorbing

e regular

5 **Conceptual:** What happens to a regular Markov chain with transition matrix $\boldsymbol{T}$ as n tends to infinity?

Answer: For a regular Markov chain with transition matrix $\boldsymbol{T}$ as n tends to infinity the matrix $\boldsymbol{T}^n$ tends to a matrix in which all the columns have equal entries.

Investigation 11

Conceptual understanding:

The powers of the transition matrix may be used to make predictions about the state of a population in the short and long term.

1. Using technology to complete the table.
2. **Factual:** What are the converging **steady state** patterns in your results?

   Answer: as the power n increases, $\boldsymbol{T}^n$ converges. Also S_n converges.

3. **Conceptual:** Generalise these convergent patterns using the terms **long-term probability matrix** and **steady state vector**.

   Answer: You can identify these convergences:

   - $\boldsymbol{T}^n \to X$ as $n \to \infty$. $\boldsymbol{X}$ is called the **long-term probability matrix**.
   - As a consequence, the matrix $\boldsymbol{X}$ satisfies the equation $\boldsymbol{TX} = \boldsymbol{X}$.
   - $S_n \to \vec{v}$ as $n \to \infty$. $\vec{v}$ is called the **steady state vector**.
   - As a consequence, the vector $\vec{v}$ satisfies the equation $T\vec{v} = \vec{v}$ as $n \to \infty$ and $XS_0 = \vec{v}$.

4. You can make interpretations such as: In the long run, the market share, which began as 81 Cike and 45 Popsi, is predicted to be 56 Cike and 70 Popsi.
5. This assumes that the population of 126 is fixed, and that the buying habits of the 126 customers are represented by the same transition matrix throughout the time taken to establish the steady state matrix.
6. **Conceptual:** How can we use powers of the transition matrix to make predictions?

   Answer: The powers of the transition matrix may be used to make predictions about the state of a population in the short and long term.

Investigation 12

Conceptual understanding:

The steady state vector $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ can be found either by finding the steady state transition matrix through considering high powers of $\boldsymbol{T}^n$ and then multiplying this by the initial distribution vector, or by solving the equations $\boldsymbol{Tu} = \boldsymbol{u}$, and $u_1 + u_2 = p$ where p is the total initial population.

1

S_0	**Total population (p) represented by S_0**	$XS_0 = \boldsymbol{v}$	$\boldsymbol{u} = \frac{1}{p}\boldsymbol{v}$
$\begin{pmatrix} 63 \\ 18 \end{pmatrix}$	81	$\begin{pmatrix} 27 \\ 54 \end{pmatrix}$	
$\begin{pmatrix} 568 \\ 123 \end{pmatrix}$	691	$\begin{pmatrix} 230.33 \\ 460.667 \end{pmatrix}$	$\frac{1}{691}\begin{pmatrix} 230.33 \\ 460.667 \end{pmatrix} = \begin{pmatrix} 0.333 \\ 0.667 \end{pmatrix}$
$\begin{pmatrix} 56358 \\ 2 \end{pmatrix}$	56360	$\begin{pmatrix} 18786.7 \\ 37573.3 \end{pmatrix}$	$\frac{1}{56360}\begin{pmatrix} 18786.7 \\ 37573.3 \end{pmatrix} = \begin{pmatrix} 0.333 \\ 0.667 \end{pmatrix}$

…

3 **Factual:** Does the steady state probability vector depend on the initial state vector?

Answer: No.

4 You can conjecture that the entries in $\boldsymbol{u} = \frac{1}{p}\boldsymbol{v}$ are also found in the columns of the steady state matrix $\boldsymbol{X}$.

5 The first two equations come from expanding

$$\begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

The third equation comes from the fact that the probabilities need to total to 1.

Simplifying either of the first two equations gives $0.4u_1 - 0.2u_2 = 0$. Solving this and $u_1 + u_2 = 1$

gives $u_1 = \frac{1}{3}, u_2 = \frac{2}{3}$

6 The process can be repeated for any regular matrix.

7 For a 3 × 3 matrix the process is the same, except that the you can use any two of the three equations given by solving $\boldsymbol{TU} = \boldsymbol{u}$ along with $u_1 + u_2 + u_3 = 1$.

8 **Factual:** What do you notice?

Answer: the entries in the steady state vector are duplicated in the columns of the steady state matrix.

9 **Conceptual:** How can you find the steady state probabilities? Give two different methods.

Answer: The steady state vector $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ can be found either by finding the steady state transition matrix through considering high powers of $\boldsymbol{T}^n$ and then multiplying this by the initial distribution vector, or by solving the equations $\boldsymbol{Tu} = \boldsymbol{u}$, and $u_1 + u_2 = p$ where p is the total initial population.

9.7 Eigenvalues and eigenvectors

Investigation 13

i Sample set of 5 pre-image points on the line $y = 2x$ $\{(-1,-2),(0.5,1),(0,0),(1,2),(2,4)\}$.

The corresponding image points are given by (6, 12)

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} -1 & 0.5 & 0 & 1 & 2 \\ -2 & 1 & 0 & 2 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 1.5 & 0 & 3 & 6 \\ -6 & 3 & 0 & 6 & 12 \end{pmatrix}.$$

The image points satisfy $y = 2x$ and thus must be on the same line. Considering the general pre-image point $(x, 2x)$ then the image point is given by $\begin{pmatrix}1 & 1\\ -2 & 4\end{pmatrix}\begin{pmatrix}x\\ 2x\end{pmatrix} = \begin{pmatrix}3x\\ 6x\end{pmatrix}$. Since $(3x, 6x)$ satisfies $y = 2x$ then the image point will always be mapped back to the line.

ii $\begin{pmatrix}1 & 1\\ -2 & 4\end{pmatrix}\begin{pmatrix}x\\ 2x\end{pmatrix} = \begin{pmatrix}3x\\ 6x\end{pmatrix} = 3\begin{pmatrix}x\\ 2x\end{pmatrix}$ and thus $\lambda = 3$.

T transforms the image points by a stretch of scale factor 3.

iii $\begin{pmatrix}1 & 1\\ -2 & 4\end{pmatrix}\begin{pmatrix}x\\ x\end{pmatrix} = \lambda\begin{pmatrix}x\\ x\end{pmatrix} \cdot \begin{Bmatrix} x + x = \lambda x\\ -2x + 4x = \lambda x\end{Bmatrix} \cdot 2x = \lambda x \therefore \lambda = 2$

TOK

The multi-billion dollar eigenvector.

Google's success derives in large part from its PageRank algorithm, which ranks the importance of webpages according to an eigenvector of a weighted link matrix

How ethical is it to create mathematics for financial gain?

Analysis of the PageRank formula provides a wonderful applied topic for linear algebra.

You might research the "$25 billion dollar eigenvector".

Should mathematics be created for need or profit?

Investigation 14

1

$$\begin{vmatrix}(5-\lambda) & -3\\ -6 & (2-\lambda)\end{vmatrix} = 0 \cdot \lambda^2 - 7\lambda - 8 = 0 \cdot (\lambda - 8)(\lambda + 1) = 0 \quad \therefore \lambda_1 = -1 \text{and} \lambda_2 = 8$$

$$\lambda_1 = -1 \cdot \begin{pmatrix}5 & -3\\ -6 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = -\begin{pmatrix}x\\ y\end{pmatrix} \cdot \begin{Bmatrix} 5x - 3y = -x\\ -6x + 2y = -y\end{Bmatrix} \cdot y = 2x. \ \therefore \boldsymbol{X}_1 = \begin{pmatrix}1\\ 2\end{pmatrix}$$

$$\lambda_1 = 8 \cdot \begin{pmatrix}5 & -3\\ -6 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 8\begin{pmatrix}x\\ y\end{pmatrix} \cdot \begin{Bmatrix} 5x - 3y = 8x\\ -6x + 2y = 8Y\end{Bmatrix} \cdot y = -x. \ \therefore \boldsymbol{X}_1 = \begin{pmatrix}1\\ -1\end{pmatrix}$$

2 a

$$\boldsymbol{AP} = \begin{pmatrix}5 & -3\\ -6 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix} = \begin{pmatrix}-1 & 8\\ -2 & -8\end{pmatrix} \qquad \boldsymbol{PD} = \begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 8\end{pmatrix} = \begin{pmatrix}-1 & 8\\ -2 & -8\end{pmatrix}$$

$\therefore \boldsymbol{AP} = \boldsymbol{PD}$

b From the definition of eigenvectors and eigenvalues we have $\boldsymbol{Ax}_1 = \lambda_1\boldsymbol{x}_1$ and $\boldsymbol{Ax}_2 = \lambda_2\boldsymbol{x}_2$. These can be combined to give ***AP*** = ***PD***.

3 a

$$\boldsymbol{AP} = \boldsymbol{PD}$$
$$\boldsymbol{APP}^{-1} = \boldsymbol{PDP}^{-1}$$
$$\boldsymbol{AI} = \boldsymbol{PDP}^{-1}$$
$$\therefore \boldsymbol{A} = \boldsymbol{PDP}^{-1}$$

b

$$\begin{aligned} \boldsymbol{A}^3 &= \left(\boldsymbol{PDP}^{-1}\right)^3 = \left(\boldsymbol{PDP}^{-1}\right)\left(\boldsymbol{PDP}^{-1}\right)\left(\boldsymbol{PDP}^{-1}\right) \\ &= \boldsymbol{PD}\left(\boldsymbol{P}^{-1}\boldsymbol{P}\right)\boldsymbol{D}\left(\boldsymbol{P}^{-1}\boldsymbol{P}\right)\boldsymbol{DP}^{-1} \\ &= \boldsymbol{PD}\left(\boldsymbol{I}\right)\boldsymbol{D}\left(\boldsymbol{I}\right)\boldsymbol{DP}^{-1} \\ &= \boldsymbol{PDDDP}^{-1} \\ \boldsymbol{A}^3 &= \boldsymbol{PD}^3\boldsymbol{P}^{-1} \end{aligned}$$

c and **d** The associative properties and the properties of the inverse are highlighted in the proofs above.

4 a

$$\boldsymbol{D} = \begin{pmatrix} -1 & 0 \\ 0 & 8 \end{pmatrix} \Rightarrow \boldsymbol{D}^3 = \begin{pmatrix} -1 & 0 \\ 0 & 512 \end{pmatrix}$$

4 b

$$\boldsymbol{D}^n = \begin{pmatrix} (-1)^n & 0 \\ 0 & 8^n \end{pmatrix}$$

5 a $\boldsymbol{A}^3 = \boldsymbol{PD}^3\boldsymbol{P}^{-1}$

$$= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 8 \end{pmatrix}^3\begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 512 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix}\left(\frac{1}{3}\right)$$

$$= \begin{pmatrix} -1 & 512 \\ -2 & -512 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix}\left(\frac{1}{3}\right)$$

$$= \begin{pmatrix} -1023 & 513 \\ 1026 & -510 \end{pmatrix}\left(-\frac{1}{3}\right)$$

$$= \begin{pmatrix} 341 & -171 \\ -342 & 170 \end{pmatrix}$$

$$\boldsymbol{A}^n = \boldsymbol{PD}^n\boldsymbol{P}^{-1}$$

b

$$= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix}^n\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} (-1)^n & 0 \\ 0 & 8^n \end{bmatrix}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}^{(-\frac{1}{3})}$$

$$= \begin{bmatrix} (-1)^n & 8^n \\ 2(-1)^n & -8^n \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}^{(-\frac{1}{3})}$$

$$= \begin{bmatrix} (-1)^{n+1} - 2\times 8^n & (-1)^{n+1} + 8^n \\ 2(-1)^{n+1} + 2\times 8^n & 2(-1)^{n+1} - 8^n \end{bmatrix}^{(-\frac{1}{3})}$$

Investigation 15

1 If a person is in town A the probability they move to town B is 0.08, and hence the probability they remain is 0.92. Similarly for those people currently in town B.

2 Population of A is 27 964. Population of B is 41 736.

3 Eigenvalues are 1 and 0.8. Eigenvectors are $\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

4 $\boldsymbol{T}^{-1} = \frac{1}{-5}\begin{pmatrix} -1 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 0.2 & 0.2 \\ 0.4 & -0.6 \end{pmatrix}$

Hence result.

5 As $n \to \infty$

$$\boldsymbol{D}^n \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence

$$\boldsymbol{T}^n \to \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0.2 & 0.2 \\ 0.4 & -0.6 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0.2 & 0.2 \\ 0.4 & -0.6 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}$$

6 The long-term populations will be

$$\begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}\begin{pmatrix} 24500 \\ 45200 \end{pmatrix} = \begin{pmatrix} 41820 \\ 27880 \end{pmatrix}$$

7 $\boldsymbol{T}^n S_0 = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0.8^n \end{pmatrix}\begin{pmatrix} 0.2 & 0.2 \\ 0.4 & -0.6 \end{pmatrix}\begin{pmatrix} 24500 \\ 45200 \end{pmatrix}$

$$= \begin{pmatrix} 41820 - 17320\times 0.8^n \\ 27880 + 17320\times 0.8^n \end{pmatrix}$$

Hence population after n years in neighborhood A is 41 820 – 17 320 × 0.8^n

and in neighborhood B is 27 880 + 17 320 × 0.8^n.

8 In 2020 n = 5 hence population in neighborhood A is 22 205.

MarComm Phones and Markov Chains

Approaches to Learning/learner profile: Communication, Critical Thinking

Exploration Criteria: Mathematical Communication (B); Use of Mathematics (E)

IB Topic: Markov Chains, Matrices

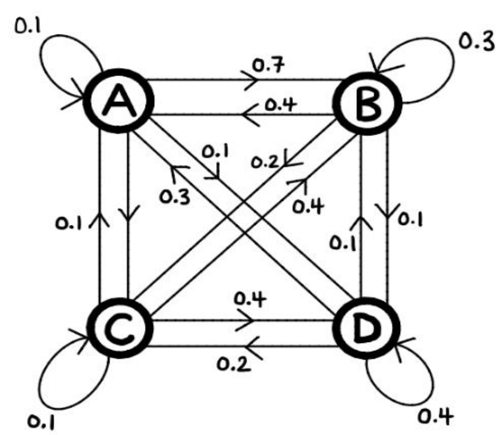
Introduction

In this task the students are looking at a (potentially) real-life example solving a problem involving Markov chains. There are brief opportunities to reflect on how the required information could be found and how reliable it may be. The main focus however, beyond practising the methods discussed in the chapter is to consider the meaning of "sophistication" and "rigor" as they are used in the exploration criterion for Use of Mathematics (E). Students look at different approaches to the same problem and consider which may be the one in which sophistication could be demonstrated. Sophistication is being able to show an understanding as well as using challenging Mathematical concepts.

The problem

A phone company, MarComm, has 4 package options, namely A, B, C and D.

A customer is initially allocated to package A for a year. After this, every year, customers can either stick with their present option or change to one of the others. The probabilities of each possible change are given in the transition state diagram



How would MarComm be able to determine the probabilities on the diagram?

Represent this information in a transition matrix, ***P***, of the system.

$$\boldsymbol{P} = \begin{pmatrix} 0.1 & 0.4 & 0.1 & 0.3 \\ 0.7 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.4 & 0.4 \end{pmatrix}$$

$P(i,j)$ gives the probability of moving from state i to state j.

If customers are all initially allocated to option A for the first year then write an initial state vector,

$\vec{x}_0$ in the form $\vec{x}_0 = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$

The probabilities for the next state can then be found by multiplying $\vec{x}_0$ by the transition matrix ***P***.

Calculate the probabilities of being in each option after 1 year if the customer starts with option A.

Write in the form $\vec{x}_1 = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$.

How might MarComm find this information useful?

$$. = \begin{pmatrix} 0.1 & 0.4 & 0.1 & 0.3 \\ 0.7 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.4 & 0.4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.7 \\ 0.1 \\ 0.1 \end{pmatrix}$$ would be the probabilities after 1 year.

The company could use the information to ensure that sufficient resources are in place to deal with each package option

Repeating this iteratively will give the probability of a customer being in any state in subsequent years.

$$\vec{x}_{n+1} = \begin{pmatrix} 0.1 & 0.4 & 0.1 & 0.3 \\ 0.7 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.4 & 0.4 \end{pmatrix}\vec{x}_n$$

which inductively gives a general result for the nth year.

$$\vec{x}_n = \begin{pmatrix} 0.1 & 0.4 & 0.1 & 0.3 \\ 0.7 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.4 & 0.4 \end{pmatrix}^n \vec{x}_0 = P^n \vec{x}_0$$

Investigate the behaviour of the system over time.

Let $\boldsymbol{\pi}$ be the stationary distribution of the system where $\boldsymbol{\pi} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix}$.

In this task you will now find the stationary distribution in three different ways.

$\boldsymbol{\pi}$ is the long-term probability of being in each of the states and is defined as

$$\boldsymbol{\pi} = \lim_{n\to\infty} \begin{pmatrix} 0.1 & 0.4 & 0.1 & 0.3 \\ 0.7 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.4 & 0.4 \end{pmatrix}^n \vec{x}_0$$

Using technology

Calculate $\vec{x}_{19}$ and $\vec{x}_{20}$ (that is, $\vec{x}_0$ and $P^{20}\vec{x}_0$) using a calculator or computer.

This will give an **indication** of the convergence to a stationary distribution (to a reasonable degree of accuracy).

Why would this not be considered a sophisticated approach to the problem?

$$\vec{x}_{19} = \begin{pmatrix} 0.1 & 0.4 & 0.1 & 0.3 \\ 0.7 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.4 & 0.4 \end{pmatrix}^{19} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.25486\ldots \\ 0.37564\ldots \\ 0.15865\ldots \\ 0.21085\ldots \end{pmatrix},$$

$$\vec{x}_{20} = \begin{pmatrix} 0.1 & 0.4 & 0.1 & 0.3 \\ 0.7 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.4 & 0.4 \end{pmatrix}^{20} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.25486\ldots \\ 0.37564\ldots \\ 0.15865\ldots \\ 0.21085\ldots \end{pmatrix},$$

This gives an indication of convergence to the stationary distribution of

$$\boldsymbol{\pi} = \lim_{n\to\infty} \begin{pmatrix} 0.1 & 0.4 & 0.1 & 0.3 \\ 0.7 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.4 & 0.4 \end{pmatrix}^{n} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.2549 \\ 0.3756 \\ 0.1586 \\ 0.2109 \end{pmatrix}, \text{ to 4 sig. figs.}$$

Solving a system of equations

This uses the fact that once the stationary distribution, π, is reached multiplying by the transition matrix has no effect $\Rightarrow \boldsymbol{P\pi} = \boldsymbol{\pi}$

So by letting $\boldsymbol{\pi} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix}$, we can solve the system of equations $(\boldsymbol{P})\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix}$

Use this method with the information in the example to find π and verify that the result is equivalent to above.

$$\begin{pmatrix} 0.1 & 0.4 & 0.1 & 0.3 \\ 0.7 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.4 & 0.4 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} - \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.9 & 0.4 & 0.1 & 0.3 & 0 \\ 0.7 & -0.7 & 0.4 & 0.1 & 0 \\ 0.1 & 0.2 & -0.9 & 0.2 & 0 \\ 0.1 & 0.1 & 0.4 & -0.6 & 0 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 & 0 & -1.20874 & 0 \\ 0 & 1 & 0 & -1.78155 & 0 \\ 0 & 0 & 1 & -0.75243 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \pi_1 = 1.20874\pi_4,\ \pi_2 = 1.78155\pi_4,\ \pi_3 = 0.75243\pi_4$$

However, since the probabilities must sum to 1 we also have $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$.

$$\Rightarrow \pi_1 = 0.2549,\ \pi_2 = 0.3756,\ \pi_3 = 0.1586,\ \pi_4 = 0.2109 \text{ to 4 sig. figs}$$

Does this feel more sophisticated than the process above? Why?

The depth of understanding required is greater. The student needs to go beyond just doing a repetitive or approximate method to understand the underlying mathematical structures and processes that are being used.

Using eigenvalues and eigenvectors

If the stationary distribution $\boldsymbol{\pi}$ is such that $\boldsymbol{P\pi} = \boldsymbol{\pi}$ then as we have seen in this chapter the matrix $\boldsymbol{P}$ must have an eigenvalue of 1 with corresponding eigenvector $\boldsymbol{\pi}$ scaled so that the sum of its elements is 1.

Find the eigenvalues of $\boldsymbol{P}$ from our example and the corresponding eigenvectors (using technology).

How could you demonstrate that you understand fully what you are finding here and that you understand the process that is taking place beyond just getting the correct answers?

How will you set out the mathematical calculations so that the process from posing the problem to answering the problem is logical?

$\lambda_1 = 1, \quad \lambda_2 = -0.38880, \quad \lambda_3 = 0.31342, \quad \lambda_4 = -0.02462,$

$$\vec{x}_1 = \begin{pmatrix} -0.48540 \\ -0.71542 \\ -0.30216 \\ -0.40157 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} -0.62357 \\ 0.74545 \\ -0.21592 \\ 0.09404 \end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix} -0.13876 \\ -0.64529 \\ 0.03356 \\ 0.75048 \end{pmatrix}, \quad \vec{x}_4 = \begin{pmatrix} 0.44404 \\ -0.39936 \\ -0.58906 \\ 0.544383 \end{pmatrix}$$

$\therefore \boldsymbol{\pi}$ is the eigenvector $\vec{x}_1$ scaled so the elements sum to 1 $\Rightarrow \boldsymbol{\pi} = \begin{pmatrix} 0.2549 \\ 0.3756 \\ 0.1586 \\ 0.2109 \end{pmatrix}$, to 4 sig. figs.

Sophisticated understanding can be demonstrated by considering more than just "getting the right answer". Students who are able to show the links and developments between the problem, the derivation of the equation $\boldsymbol{P\pi} = \boldsymbol{\pi}$ and the use of eigenvectors and eigenvalues in solving it will likely score highly on criterion E in an exploration. A logical development of the work with explanations of different steps throughout rather than just the steps will result in a more rigorous piece of work.

These methods, therefore, gives MarComm, $\boldsymbol{\pi}$, the probabilities of a customer being in each of the four options in the long term.

What could the company do with this information?

Clearly all three methods produce the same approximate vector $\boldsymbol{\pi}$ correct to 4 sig. figs.

However, which method do you prefer?

Which method is the most sophisticated? Justify your answers.

Clearly the last method is the most sophisticated and not just because it is the most difficult to understand potentially. It is because it links together different areas of mathematics and the underlying structures need to be identified. Students need to really be clear about what processes they are using together with why. They could of course just "plug the values into the formula", but this would not demonstrate a sophisticated level of understanding.

Extension work

Does a system always reach a steady state?

How quickly will the process converge to the steady state?

This extension work is more general rather than example specific and demonstrating understanding of this could be an indication of a very sophisticated level of mathematics – this is quite challenging. Although it is going beyond the level of the course this is not a prerequisite for sophisticated maths.

10 Analyzing rates of change: differential calculus

Essential understandings

Calculus describes rates of change between two variables and the accumulation of limiting areas. Understanding these rates of change allows us to model, interpret and analyze real-world problems and situations. Calculus helps us understand the behavior of functions and allows us to interpret the features of their graphs.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Students will understand the links between the derivative and the rate of change and interpret the meaning of this in context.
- Finding patterns in the derivatives of polynomials and their behavior, such as increasing or decreasing, allows a deeper appreciation of the properties of the function at any given point or instant.
- Optimization of a function allows us to find the largest or smallest value that a function can take in general and can be applied to a specific set of conditions to solve problems.
- Maximum and minimum points help to solve optimization problems.
- Kinematics allows us to describe the motion and direction of objects in closed systems in terms of displacement, velocity and acceleration.
- Many physical phenomena can be modelled using differential equations and analytic and numeric methods can be used to calculate optimum quantities.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
The gradient of a curve at a point is equal to the gradient of the tangent to the curve at that point. The gradient of a curve at a point can be found as the limit of the gradient of a chord from that point to a second point on the curve as the distance between them approaches 0.	Investigation 1
Depending on the domain, maximum or minimum values of a function could be the stationary points of a function called the local maximum or minimum points or the actual maximum or minimum points of the function called the global/absolute maximum or minimum.	Investigation 2
The derivatives of composite functions can be found using the chain rule.	Investigation 3
Exponential and logarithmic functions can be differentiated since the derivative of an exponential function with base e leads to itself and this produces the simplest results.	Investigation 5
A horizontal point of inflexion satisfies the conditions that the first derivative and second derivative equal zero and displays a change in concavity on either side of the stationary point.	Investigation 6

The chain rule can be applied to related rates problems which analyse the effect that the change in one rate has on another rate.	Investigation 7

Syllabus sections covered in this chapter:

- SL5.1*
- SL5.2*
- SL5.3*
- SL5.4*
- SL5.6
- SL5.7
- AHL5.9
- AHL5.10
- AHL5.12

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 427	Page 440: Example 10 Page 460: Example 21 Page 463: Example 22	Page 431: Example 2 Page 437: Example 8 Page 438: Example 9 Page 440: Example 10 Page 456: Example 19 Page 460: Example 21	Pages 441, 454, 464

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 465	Page 467	N/A

Developing inquiry skills

The aim of this opening problem is to try to mimic how differentiation might be used in a real-life situation to solve optimization problems. Hence it begins with data collection and analysis. The questions here can all be answered before the chapter begins and will provide some review for earlier work.

- Plot the data on your GDC.
- Explain why these two models might be suitable.
- Find best fitting equations for each of the models.

Answer: Both models are decreasing with an asymptote at $x = 0$, which means there will exist a price at which no-one (measured to the "nearest person") will purchase the product. This is an opportunity to discuss what an asymptote might mean when modelling discrete data.

Equations obtained by regression analysis are $d = \frac{855}{x^{1.25}}$ and $d = 130(0.924)^x$ or $d = 130\mathrm{e}^{-0.0792x}$

- The business would like to maximize their profit. What other information would they need?

Answer: They would need to know the size of the market, the costs of producing the product and whether all the items produced will be sold.

Other factors which are not needed for calculating this model but might affect the demand curve data include advertising, customer feedback, etc.

- If this information were available, how might you work out the maximum value of a profit function?

Answer: The answer that should be familiar from work in earlier chapters might be to plot the curve and find the maximum.

- The gradient of the demand curve is called the "marginal demand". What might the marginal demand tell you? How could you work out its value from the curve?

Answer: The gradient or the marginal demand will be the increase (or decrease) in demand for one unit of increase in price. This should be familiar from their work with straight lines. At this point they will not know what is meant by the gradient of a curve but can suggest that the marginal cost at a point will be the gradient of the chord between that point and the point one unit to the right.

- The marketing team has a model that links extra demand with the amount of money spent on advertising. How can this be incorporated into your model?

Answer: At this point they could try to create an equation that models total revenue less marketing costs. They will have to make assumptions such as all the product produced is also sold and how the cost of a marketing campaign might affect the demand for the product. Manufacturing costs could be included as well, later in the chapter they are given as \$3n when n items of the product are produced. There is no right answer to this question!

10.1 Limits and derivatives

Investigation 1

Conceptual understanding:

The gradient of a curve at a point is equal to the gradient of the tangent to the curve at that point.

The gradient of a curve at a point can be found as the limit of the gradient of a chord from that point to a second point on the curve as the distance between them approaches 0.

Additional notes

This investigation explores the idea of a limit and leads to the first principles formula for the derivative. It does this by first demonstrating that the gradient of the curve can be thought of as being equal to the gradient of the tangent and then showing that the gradient of the tangent is the limit of the gradient of a chord as the distance between the two points on the chord tends to zero.

The first principles formula is not required for the exam but it provides a useful understanding of where calculus comes from and is used later in the chapter to indicate how other results are derived.

It is required for the investigation that students have access to a GDC or software that will draw and give the equation of a tangent to a curve.

1 **a** From earlier chapters the students will already have an understanding of positive and negative gradients.

i $x < 0$

ii $x > 0$

iii $x = 0$

b A. Students might say B if they are thinking that "greatest" is the same as "steepest".

2 **a** All GDCs should have a function to draw a tangent; if not it can be done on Geogebra.

b **Factual:** What do you notice about the gradient of the tangent and the gradient of the curve at the point of contact?

Answer: The gradient of a curve at a point is equal to the gradient of the tangent to the curve at that point.

c

x	−1	0	1	2	3
Gradient of tangent at x	−2	0	2	4	6

d This is just an exercise in pattern spotting so they should see gradient = $2x$.

3 **a** The gradient of the chord gets closer to the gradient of the tangent.

b The gradient of the chord will equal the gradient of the tangent.

4 **a** Gradient $= \dfrac{\text{Change in } y \text{ value}}{\text{Change in } x \text{ value}} = \dfrac{(x+h)^2 - x^2}{x+h-x} = \dfrac{(x+h)^2 - x^2}{h}$

b You would have $\frac{0}{0}$.

c

$$\frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h \to 2x$$

as $h \to 0$.

d This is the same as the answer to **2d** which justifies the conjecture made there.

5 This repeats the process of the earlier questions. The pattern $3x^2$ will be slightly harder to spot. If they cannot do so they can move to part **c** and see if the result they obtain gives them an expression that fits the pattern.

a

x	−2	−1	0	1	2	3
Gradient of tangent at *x*	12	3	0	3	12	27

b gradient = $3x^2$

c $\dfrac{(x+h)^3 - x^3}{h}$

d $\dfrac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = 3x^2 + 3xh + h^2 \to 3x^2$

as $h \to 0$.

This should be the same as they conjectured in part **b**

6 **Conceptual:** How would you find the gradient of the curve at a point using the gradient of a chord from that point?

Answers: The gradient of a curve at a point can be found as the limit of the gradient of a chord from that point to a second point on the curve as the distance between them approaches 0.

TOK

What value does the knowledge of limits have?

Is infinitesimal behavior applicable to real life?

Are intuition and imagination valid ways of knowing in mathematics?

TOK

Mathematics and the real world: the seemingly abstract concept of calculus allows us to create mathematical models that permit human feats, such as getting a man on the Moon. What does this tell us about the links between mathematical models and physical reality?

To explore the relation between mathematical models and reality, the essential issue of mathematical modelling is dependent on social and personal construction processes where absolute agreement cannot be expected.

A prime pragmatic function of such models is to enable calculations and predictions of physical phenomena. Mathematical models are useful also in representing aspects of reality that are hard to visualize. Models serve as conceptual frameworks that can lead to important physical discoveries.

A counter-claim would be that a variety of different mathematical models can account for the same appearances, for example, the models of Ptolemy and Copernicus. This implies that the construction of models involves a substantial element of creative imagination.

Investigation 2

Conceptual understanding:

Depending on the domain, maximum or minimum values of a function could be the stationary points of a function called the local maximum or minimum points or the actual maximum or minimum points of the function called the global/absolute maximum or minimum.

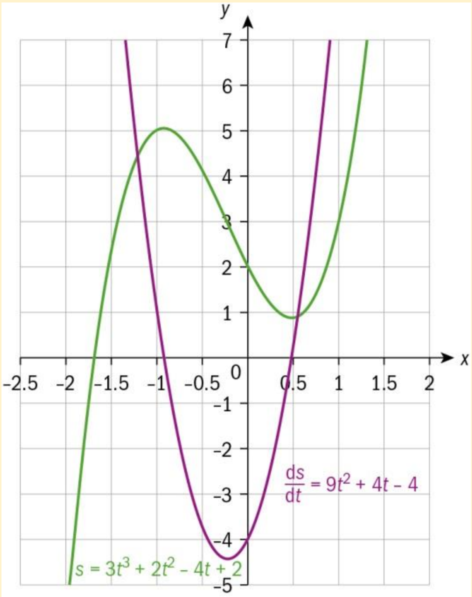
Additional notes

The purpose of the investigation is to demonstrate the use of differentiation to find local maximum and minimum points and to make students aware that these might not be the same as the global maximum or minimum points.

1 $\dfrac{ds}{dt} = 9t^2 + 4t - 4$

Using the usual rules for differentiating.

2 There are two ways of doing this question either plotting the curve found in part **1** or use the derivative function on the GDC. This is a technique that is expected for the exam and this would be a good opportunity to introduce this technique if not done so already. Before moving onto the next question it would be best if they hid the graph of $s(t)$ so the focus can be on the derivative graph.



3 The equation is solved by finding the zeros for the equation for the derivative or by solving $9t^2 + 4t - 4 = 0$ and then substituting to find the s-values (−0.925, 5.04), (0.481, 0.873).

Do not at this stage take the short cut of finding the local maximum and minimum points.

4 At these values of t the curve has a turning point: one is a local maximum point and the other a local minimum point.

This observation introduces the alternative method for solving **3**, namely using the GDC to find the maximum and minimum points on the original curve. The fact it gives the same results could be demonstrated here.

It is perhaps worth talking about the two techniques. Generally in an exam if given the original function finding the maximum or minimum point is better, if given the derivative function then finding its zeros is more useful.

5 Again ask them to hide the graph of $s(t)$ so the focus is on the graph of the derivative.

The value of $\frac{ds}{dt}$ goes from negative to positive for a minimum point and from positive to negative for a maximum point.

6 26 and −6

The maximum and minimum values (sometimes called the **global** maximum and minimum) are at the end points of the curve. Points where the gradient is equal to zero are referred to as local maximum and minimum points to indicate that they might not actually be the highest/lowest values on the function.

7 **Conceptual:** On a function with a restricted domain where might the maximum and minimum points occur?

Answer: Depending on the domain, maximum or minimum values of a function could be the stationary points of a function called the local maximum or minimum points or the actual maximum or minimum points of the function called the global/absolute maximum or minimum.

Developing inquiry skills

Given the cost ($\$C$) of producing n items is $C = 3n$ and the market size is estimated to be 10 000 people and assuming only sufficient items to match the demand are made, find an expression for the profit (P).

By differentiating this expression and solving $\frac{dP}{dx} = 0$, find the maximum profit and the price at which the product should be sold to achieve this.

Answer: The number of people expected to buy the product at $\$x$ is

$$\frac{d}{100} \times 10000 = 100d = \frac{85500}{x^{1.25}}.$$

The revenue will therefore be $100d \times x = \frac{85500}{x^{0.25}}$.

The cost is $3\left(\frac{85500}{x^{1.25}}\right)$, as number produced is assumed equal to demand.

The profit equation is revenue minus costs and so $P = \frac{85500}{x^{0.25}} - 3\left(\frac{85500}{x^{1.25}}\right)$.

Though it seems complicated the solving the equation is relatively straightforward.

$$\frac{dP}{dx} = -\frac{21375}{x^{1.25}} + \frac{320625}{x^{2.25}} = 0$$

$$x = \frac{320625}{21375} = 15$$

They should sell the product for \$15 and would expect to make approximately \$34 760.

10.2 Differentiation: further rules and techniques

TOK

Mathematics: invented or discovered?

If mathematics is created by people, why do we sometimes feel that mathematical truths are objective facts about the world rather than something constructed by human beings?

Ask students to write down three things that were invented and three things that were discovered.

Inventions might include the airplane, lightbulbs, computer and discoveries might include dinosaur fossils, magnets, the source of the Nile.

Ask students to create and share their own definitions.

We are now approaching one of the big TOK questions; is mathematics discovered or invented?

An invention is something that was not previously there. A discovery concerns something that already exists at the time of discovery but was previously unknown. As a result of the discovery, nothing has changed apart from an associated increase in knowledge.

Is there something that is in the intersection? What about music?

Investigation 3

Conceptual understanding:

The derivatives of composite functions can be found using the chain rule.

Additional notes

The investigation will lead to the use of the chain rule to differentiate composite functions.

Some revision of composite functions might be required.

1 $y = 4x^4 - 20x^2 + 25$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 16x^3 - 40x = 8x\left(2x^2 - 5\right)$$

2 **a** $y = u^2$

b $\frac{\mathrm{d}y}{\mathrm{d}u} = 2u,\ \frac{\mathrm{d}u}{\mathrm{d}x} = 4x$

c $\frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 2u \times 4x = 8x\left(2x^2 - 5\right)$

3 The two answers are equal.

4 **a** $y = x^3 + 3x^2 + 3x + 1$, $\frac{dy}{dx} = 6x^5 - 6x^2$

b $u = x^3 - 1$

c $y = u^2$

d $2u \times 3x^2 = 6x^2\left(x^3 - 1\right)$

e The functions on parts **a** and **d** are equal.

5 Hopefully enough has been done by this point for the students to be able to spot the formula

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

6 **a** Expansion is $y = x^2 + 4 + 4x^{-2}$

$$\frac{dy}{dx} = 2x - 8x^{-3}$$

b $u = x + 2x^{-1} \Rightarrow \frac{du}{dx} = 1 - 2x^{-2}$

$$y = u^2 \Rightarrow \frac{dy}{du} = 2u$$

$$\frac{dy}{du} \times \frac{du}{dx} = 2u \times \left(1 - 2x^{-2}\right) = 2\left(x + 2x^{-1}\right)\left(1 - 2x^{-2}\right)$$

$$= 2\left(x - 2x^{-1} + 2x^{-1} - 4x^{-3}\right) = 2\left(x - 4x^{-3}\right)$$

This is the same as the expression in **a** so the conjecture is verified.

7 **Conceptual:** How do you find the derivative of a composite function?

Answer: The derivatives of composite functions can be found using chain rule **1–4**.

TOK

What is the difference between inductive and deductive reasoning?

Deductive reasoning usually follows steps. A suitable definition is that deductive reasoning is the process by which a person makes conclusions based on previously known facts.

You might want to give examples such as "all dolphins are mammals, all mammals have kidneys; therefore, all dolphins have kidneys", or $x = y$ and $y = z$, therefore $x = z$.

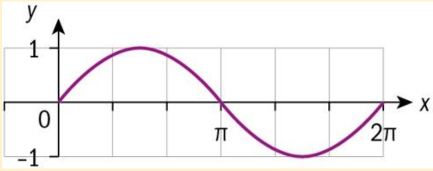
Inductive reasoning is the opposite of deductive reasoning. Inductive reasoning draws conclusions based on a set of observations. This might not be a valid method of proof. Just because you observe a number of situations in which a pattern exists doesn't mean that that pattern is true for all situations.

Examples such as "60% of the students in your class like strawberries, so 60% of people in the world like strawberries", or "Ali's mother and father are doctors, Ali is a doctor, so Ali's brother must be a doctor".

Investigation 4

This investigation leads to the discovery of the rules for differentiating trigonometric functions. A proof using limits involves use of the compound angle formula so could be shown to students if they are aware of these identities.

1 a



b The gradient at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ are both equal to 0. The gradient at 0 and 2π is 1 and the gradient at π is −1. When estimating from the graph the units used would have to be radians and the scales of both axes would have to be equal for the correct result to be estimated.

The purpose of this question is to emphasize the link between gradients and derivatives, so that the numbers generated in question **2** have some context.

2 If the student's GDC has the capacity to enter the derivative of a function this table can be generated using the table function with a step length of $\frac{\pi}{4}$. In this case the graph of the derivative of sin$x$ can also be overlaid with the graph of cos$x$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\sin x$	0	0.707	1	0.707	0	−0.707	−1	−0.707	0
$(\sin x)'$	1	0.707	0	−0.707	−1	-0.707	0	0.707	1
$\cos x$	1	0.707	0	−0.707	−1	−0.707	0	0.707	1
$(\cos x)'$	0	−0.707	−1	-0.707	0	0.707	1	0.707	0

3 $\frac{d(\sin x)}{dx} = \cos x\,,\ \frac{d(\cos x)}{dx} = -\sin x$

4 a 0.0175

b −0.0175

c The scale on the x-axis has changed so the gradients of the tangents are different. This implies that the values of the derivatives of the sin$x$ are no longer equal to the values of cos$x$.

5 $\tan x = \frac{\sin x}{\cos x}$ so $\frac{d(\tan x)}{dx} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

6 Factual: What units do you use when differentiating trigonometric functions?

Answer: You need to use radians in order to have the derivatives in the simplest possible form.

7 The cosine function describes the gradient of the tangent lines to the graph of the sine function for each value in the domain of the sine function in radians.

In this case the derivatives of $\sin x, \cos x$, and $\tan x$ are $\cos x, -\sin x$ and $\frac{1}{\cos^2 x}$.

TOK

Euler was able to make important advances in mathematical analysis before calculus had been put on a solid theoretical foundation by Cauchy and others. However, some work was not possible until after Cauchy's work.

What does this suggest regarding intuition and imagination in mathematics?

This concerns knowledge claims in mathematics. You might want to consider a few different ways of knowing with statements such as:

- Reason and imagination are equally important to understanding mathematics. If one can only visualize it in the head but cannot express it satisfactorily, then the conclusion is flawed and inconsistent.
- Much work can also be done using intuition if one has an ingenious insight of the material. For instance, Einstein knew the big ideas of his general theory of relativity but he just lacked the necessary mathematical language to present it until he was helped by mathematicians such as Sir Arthur Eddington.

Investigation 5

Conceptual understanding:

Exponential and logarithmic functions can be differentiated since the derivative of an exponential function with base e leads to itself and this produces the simplest results.

Additional notes

The investigation will lead to the discovery of the rules for the derivatives of $y = \mathrm{e}^x$ and $y = \ln x$ by deducing a pattern in the values of the gradients at certain points on the curves.

1 Both functions are exponential. Their gradients seem to be exponential also (but because they do not cross the y-axis at 1 then there must be a vertical scaling factor).

For $y = 2^x$ the gradient is always less than the value of the function and for $y = 3^x$ it is always greater than the value of the function.

2 Such a function might be of the form $y = a^x$ where $2 < a < 3$.

3 **a** On a suitable GDC the two functions could be put in adjacent columns of a table function.

x	−1	0	1	2	3
e^x	0.368	1	2.718	7.389	20.086
$\left(\mathrm{e}^x\right)'$	0.368	1	2.718	7.389	20.086

b $\dfrac{\mathrm{d}\left(\mathrm{e}^x\right)}{\mathrm{d}x} = \mathrm{e}^x$

4 A sketch should be a decreasing function in the first quadrant with a vertical asymptote at $x = 0$, and a horizontal asymptote at $y = 0$.

The sketch will help provide justification for the pattern spotted in the next part.

5 a

x	1	2	3	4	5
$(\ln x)'$	1	0.5	0.333...	0.25	0.2

b i $\dfrac{\mathrm{d}(\ln x)}{\mathrm{d}x} = \dfrac{1}{x}$

ii The sketch should resemble the graph of $y = \frac{1}{x}$.

iii When x = 10 gradient is 0.1, hence verifying the conjecture.

6 Factual: What base should be used when differentiating exponential and logarithmic functions.

Answer: When differentiating exponential and logarithmic functions base e produces the simplest results.

7 Conceptual: Why is the exponential function with base e so special?

Answer: Exponential and logarithmic functions can be differentiated since the derivative of an exponential function with base e leads to itself and this produces the simplest results.

TOK

Who do you think should be considered the discoverer of calculus?

The debate over whether Newton or Leibnitz discovered certain calculus concepts.

An opportunity to have students research and present a document/wall display.

Instructions to students might include:

- I think it was Newton or Leibnitz.
- You find one more person from a calculus timeline.
- Take on the role of their representative and in one paragraph, state their case with reasons and evidence.
- Now take on the role of the judge and in one paragraph, stet who you think should be called the discoverer of calculus and why.

You might want a picture of the mathematicians with their paragraphs and a picture of the student with their summary. This will see students taking ownership of their decisions that might even be called "risk taking".

Developing inquiry skills

This return to the problem seeks to find the maximum profit using the second of the models, namely $d = 130\mathrm{e}^{-0.0792x}$. It also deals with an optimization question in which a formula needs to be constructed using two variables.

Given the cost (\$$C$) of producing n items is $C = 3n$ and the market size is estimated to be 10 000 people, and assuming only sufficient items to match the demand are made,

1 find an expression for the profit (P)

Answer: The number of people expected to buy the product at \$$x$ is

$$\frac{d}{100} \times 10{,}000 = 100d = 13000e^{-0.0792x}$$

The revenue will therefore be $100d \times x = 13\,000xe^{-0.0792x}$

The cost is $3(13\,000e^{-0.0792x})$, as number produced is assumed equal to demand.

The profit equation is revenue minus costs and so

$$P = 13000xe^{-0.0792x} - 39000e^{-0.0792x} = 13000e^{-0.0792x}(x-3)$$

2 by differentiating this expression and solving $\frac{dP}{dx} = 0$, find the maximum profit and the price at which the product should be sold to achieve this.

Answer: Using calculus rather than a GDC

$$\frac{dP}{dx} = 13000\left(e^{-0.0792x} - 0.0792xe^{-0.0792x} + 3 \times 0.0792e^{-0.0792x}\right)$$

$$= 13000e^{-0.0792x}(1 - 0.0792x + 0.2376) = 13000e^{-0.0792x}(1.2376 - 0.0792x) = 0$$

$x = 15.63$ and $47 600 profit.

10.3 Applications and higher derivatives

Investigation 6

Conceptual understanding:

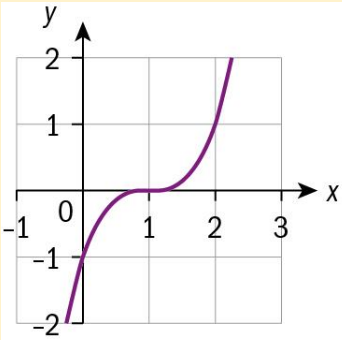
A horizontal point of inflexion satisfies the conditions that the first derivative and second derivative equal zero and displays a change in concavity on either side of the stationary point.

Additional notes

This investigation demonstrates the significance of the second derivative both as an indicator of the rate of change of the gradient and of the concavity of the curve.

1 In the first curve the gradient is increasing and in the second it is decreasing.

2 a



b $\frac{dy}{dx} = 3(x-1)^2$

c $\frac{dy}{dx} > 0$ for all values of x

3 a $\frac{d^2y}{dx^2} = 6(x-1)$

b i $x > 1$

ii $x < 1$

4 Conceptual: What feature of the graph is indicated by the sign of the second derivative?

Answer: The sign of the second derivative indicates whether a curve is concave up or concave down.

5 a $x = 1$

b The concavity changes from down to up.

6 Conceptual: Given the equation of a curve how would you find a point of inflexion if one exists?

Answer: A horizontal point of inflexion satisfies the conditions that the first derivative and second derivative equal zero and displays a change in concavity on either side of the stationary point.

International-mindedness

The Greeks' mistrust of zero meant that Archimedes' work did not lead to calculus.

Archimedes, one of the greatest ancient Greek mathematicians of all times. Archimedes was a Greek ancient mathematician, astronomer, physicist, inventor, and engineer. He is credited with introducing infinitesimals, the foundation of calculus.

His work was not considered valid as the ancient Greeks did not have zero in their counting system and had a general mistrust of the number.

TOK

Does the fact that Leibnitz and Newton came across the calculus at similar times support the argument of Platonists over Constructivists

Here we are looking at the nature of mathematics.

Platonists would say that mathematical objects exist independently of the human mind and are, thus, discovered. This is often claimed to be the view most people have of numbers.

This objective existence, however, does not mean an empirical existence but, rather, an abstract existence, hence its 'Platonic' label.

The main problem with a Platonistic view of mathematics is an epistemic one. If, indeed, mathematical objects are abstract objects and objectively exist, then how do we know anything about them?

Constructive mathematics requires that proof be algorithmic. The emphasis in constructive theory is placed on hands-on provability, instead of on an abstract notion of truth.

Note that these are not the only two philosophies of mathematics.

Investigation 7

Conceptual understanding:

The chain rule can be applied to related rates problems which analyse the effect that the change in one rate has on another rate.

Additional notes

This investigation shows how the chain rule can be used to link related rates.

1 The radii will be increasing most quickly for smaller values of r and hence when $t = 1$. This can be verified once the actual values have been calculated.

2 $\frac{dr}{dt}$ is the most useful version for this investigation though r' and $\dot{r}$ are also possible.

3 $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

4 As $\frac{dV}{dt}$ is known it is only necessary to find $\frac{dV}{dr}$.

5 $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$

Hence

$$10 = 4\pi r^2 \times \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{10}{4\pi r^2}$$

6 $0.796\ \text{m s}^{-1}$ and $0.0884\ \text{m s}^{-1}$

7 **Conceptual:** How can you find an expression connecting two related rates?

Answer: The chain rule can be applied to related rates problems which analyse the effect that the change in one rate has on another rate.

TOK

How can you justify the raise in tax for plastic containers, e.g. plastic bags and plastic bottles, using optimization?

An important environmental concern that is well document in the media where students might research and write a report using a mathematical model which would allow them to access the skills of mathematical presentation, communication and personal engagement with areas such as the countries where customers have to pay for plastic supermarket bags or a tax increase on water sold in plastic bottles.

Developing inquiry skills

1 Explain why the derivative of the profit function with respect to the number of goods produced is often used as an approximation for marginal profit.

Answer: The marginal profit is the extra profit when increasing production by one unit. This is equivalent to the gradient of the chord joining the cost of production to the point on the curve one unit on. If there are many units then the gradient of this chord is likely to be close to the gradient of the tangent of the curve. The derivative is simpler to calculate and clearly shows the trends of the marginal profit.

2 Use related rates to find an expression for the marginal cost, $\frac{dP}{dd}$ in terms of d.

Answer: By the chain rule

$$\frac{dP}{dd} = \frac{dP}{dx} \times \frac{dx}{dd}$$

$$P = 13000e^{-0.0792x}(x-3)$$

$$\frac{dP}{dx} = 13000 \times -0.0792e^{-0.0792x}(x-3) + 13000e^{-0.0792x} \times 1$$

$$= 13000e^{-0.0792x}(-0.0792x + 3 \times 0.0792 + 1)$$

$$= 13000e^{-0.0792x}(-0.0792x + 1.2376)$$

$$\frac{dd}{dx} = 13000 \times -0.792e^{-0.792x}$$

Hence

$$\frac{dP}{dd} = 13000e^{-0.0792x}(-0.0792x + 1.2376) \times \frac{1}{13000 \times -0.0792e^{-0.0792x}}$$

$$-\frac{1}{0.0792}(-0.0792x + 1.2376) = x - 15.63$$

Because the result is required in terms of d the demand we use $d = 13000e^{-0.0792x}$ replace x with

$$x = -\frac{1}{0.0792}\ln\left(\frac{d}{13000}\right)$$

to get

$$\frac{dP}{dd} = -\frac{1}{0.0792}\ln\left(\frac{d}{13000}\right) - 15.63.$$

It is worth noting that the same result could have been found by replacing x with

$$x = -\frac{1}{0.0792}\ln\left(\frac{d}{13000}\right)$$

in the original equation and so differentiating

$$P = d\left(-\frac{1}{0.0792}\ln\left(\frac{d}{13000}\right) - 3\right)$$

by the chain and product rule.

3 Find an expression for $\frac{d^2P}{dd^2}$ and hence show the marginal profit is always decreasing.

Answer:

$$\frac{d^2P}{dd^2} = -\frac{1}{0.0792} \times \frac{1}{d}$$

which is negative and so $\frac{dP}{dd}$ is always decreasing.

4 Use this rule to find the maximum value of *d* at which the factory should produce goods.

Answer: Need $\frac{dP}{dd} > 0$, hence

$$\frac{dP}{dd} = -\frac{1}{0.0792}\ln\left(\frac{d}{13000}\right) - 15.63 > 0$$

This can be solved on the GDC or rearranged to give

$$\ln\left(\frac{d}{13000}\right) < 1.24$$

$$d < 13000e^{1.24} = 3770$$

River crossing

Approaches to Learning/learner profile: Thinking Skills: Evaluate, Critiquing, Applying

Exploration Criteria: Personal Engagement (C) Reflection (D), Use of Mathematics (E)

IB Topic: < Differentiation, Optimisation

Introduction

This task introduces the students to the idea that their inspiration for an exploration can come from many different sources – one of which, in this case, may be examples and questions in textbooks. However if they use these then they are advised to contextualize that idea in real life in order to score well in criterion C – personal engagement. They also need to reflect on the occasional necessary simplifications, assumptions, guesses or estimates that are required in order to model and solve a complex situation with multiple often unknowable variables (Reflection (D)). The task also gives the student the opportunity to practise another optimization problem and demonstrate their understanding of the mathematics of this (Use of Mathematics (E)).

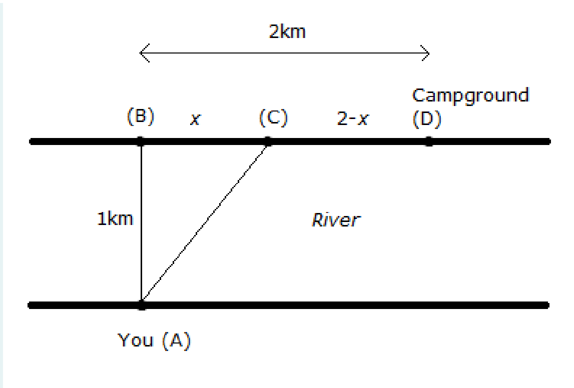
It is possible to use textbook examples and questions from exercises as inspiration or springboards for an exploration idea. If a question you have done resonates with you or reminds you of a particular experience or can be adapted, then it may be possible to develop it into a workable exploration idea.

Start by answering the following "textbook" question using the optimization techniques learned in this chapter:

The problem

You are standing at the edge of a slow-moving river which is one kilometer wide and wish to return to your campground on the opposite side of the river. You can swim at 3 km/h and run at 8 km/h. You must first swim across the river to any point on the opposite bank. From there run to the campground, which is 2 km from the point directly across the river from where you start your swim. What route will take the least amount of time?

Here is a diagram of this situation. You are at A. B is directly opposite on the other bank. C is a point on the opposite bank *x* km from B. D is the campground



What is the length of AC in terms of x?

Using Pythagoras' theorem $\text{AC} = \sqrt{x^2+1}$.

Now recall that is travel is at a constant rate of speed then

time taken $= \frac{\text{Distance travelled}}{\text{Speed}}$.

Using this formula give an expression in terms of x for the time taken for swimming from A to C.

Time taken to swim from A to C $= \frac{\sqrt{x^2+1}}{3}$. .

Also give an expression for the time taken for the running section (C to D) in terms of x.

Time taken to run from C to D $= \frac{2-x}{8}$. .

Hence give an expression for the total time taken, T, for travelling from A to D in terms of x.

Total time taken T $= \frac{\sqrt{x^2+1}}{3} = \frac{2-x}{8}$.

We wish to minimize this expression (find the minimum time taken).

First simplify

$$T = \frac{\sqrt{x^2+1}}{3} = \frac{2-x}{8}.$$

$$= \frac{1}{3}\left(x^2+1\right)^{\frac{1}{2}} + \frac{1}{4} - \frac{x}{8}.$$

Find $\frac{dT}{dx}$

$$\frac{dT}{dx} = \frac{1}{3} \times \frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}} \times 2x - \frac{1}{8}$$

$$= \frac{x}{3\sqrt{x^2+1}} - \frac{1}{8}.$$

Now solve $\frac{dT}{dx} = 0$ to determine the value of x that minimizes the time taken.

$$\frac{x}{3\sqrt{x^2+1}} - \frac{1}{8} = 0$$

$$64x^2 = 9\left(x^2+1\right)$$

$$64x^2 = 9x^2 + 9$$

$$55x^2 = 9$$

$$x^2 = \frac{9}{55}$$

$$x = \pm\frac{3}{\sqrt{55}}$$

Now $x \neq -\frac{3}{\sqrt{55}}$ as x is a length and so cannot be negative.

Therefore $x = \frac{3}{\sqrt{55}}$ = 0.405 km.

How do we know this is a valid value?

The value is less than 2. It is positive.

We could also check that it is a minimum value by using the second derivative test.

For the value of x found find the minimum time possible and describe the route.

$$T = \frac{\sqrt{\left(\frac{3}{\sqrt{55}}\right)^2 + 1}}{3} + \frac{\left(2 - \frac{3}{\sqrt{55}}\right)}{8} = 0.360 + 0.199 = 0.550 \text{ (3sf)}$$

Therefore minimum time is 0.559 hours (= 33.5 minutes).

You would swim from point A to a point C which is 0.405 km from B. From here you would run the remaining 1.595 km to the campsite. The swim would take 0.360 hours (21.6minutes) and the run would take 0.199 hours (11.9 minutes).

Many students use optimization as a basis for their exploration.

Assume that this question is based on a real-life experience where the only way to return to the campsite is by swimming across the river (there is no boat and no bridge) and running down the other side and the aim to to do the journey in as short a time as possible.

Look back at the original question.

What other assumptions have been made in the question?

- The river is exactly 1 km wide.
- The campsite is exactly 2 km along the bank on the other side.
- It is possible to swim at a constant speed of 3 km. The river flow does not push you downstream or become more difficult to maintain a speed in the middle.
- There is no time taken to enter and leave the river.
- It is possible for you to run at a constant speed of 8 km – over any terrain – for the remaining distance.
- The banks of the river are perfectly parallel and perfectly straight.

What information in the question are you unlikely to know when you are standing at the edge of the river?

- The width of the river
- The speed you can swim or run
- The distance to the campsite on the other side.

What additional information would you need to know to determine the shortest time possible?

- The effect of the flow of the river
- What the person is carrying and whether this will have any effect.

Criticize the question as much as possible!!

The question is a simplified version of a real-life situation.

In fact the question is perhaps more accurately stated as:

You are standing at the edge of a river and wish to return to your campground which you can see further down the river on the other side. You must first swim across the river to any point on the opposite bank. From there run to the campground. What route will take the least amount of time?

What methods could you use to find the width of the river from where you are standing? How accurate is your answer? How accurate does it need to be?

It is possible to use trigonometry or similar triangles if you have instruments to measure distance and angle. Trigonometry was covered in Chapter 1 and further in Chapter 11 but some students may already be familiar with using right angled triangles and trigonometry. Similar triangles are in presumed knowledge.

Students could be guided to research these methods.

It may also be possible to measure on a map if it is a reliable scale or use GPS if you have it available!

The other method is to guess or estimate based on a known distance.

The answer will have varying degrees of accuracy depending on the accuracy of the measuring instruments used or the ability to guess accurately.

Similarly how can you find how far down the river the campsite is?

As above

At what speed can you run? At what speed can you can swim?

This is most likely going to be based on a known running/swimming time for a given distance.

What assumptions are you making when you answer these questions?

As above – that this time remains constant regardless of river flow or terrain to run.

In order to make it possible to answer the question in "real life" certain assumptions, guesses and estimates do have to be made; otherwise the question is too complex to answer as it has too many unknowns and variables. However, if this were an exploration it would be important to reflect critically on any assumptions made and the subsequent significance and limitations of the results.

It is also possible to argue, perhaps, that if you had just set off and jumped in the river to swim across you would already be back in the campsite by the time you have managed to complete all these mathematical calculations!

In this chapter you have been introduced to some classic optimization problems in the examples and exercises. For example there is the "open-box problem" in Q5 of exercise 10E and example 10 involving the "volume of a cylindrical can" on page 440–442.

Now consider the open-box problem and the cylindrical can problem and/or some of the questions in Exercise 10E on pages 441–442. If you were writing an exploration and these problems were forming the basis or inspiration of that exploration then …

What assumptions have been made in the question?

What information in the question are you unlikely to know in real life?

How could you find it?

What additional information would you need to know?

Criticize the questions as much as possible!

One problem with the open-box problem is that it does not provide the box with any extra material to build stability in to the construction. A box built in such a way would not likely be strong and so It would be wise to have tabs on the sides being folded that can be used to add this stability.

In both the open-box problem and the can problem the question of aesthetics is not considered. Also functionality. Cans, for example, are not always built in such a way to provide maximum volume for the minimum use of materials but also with design, packaging and functionality (e.g. considering average grip size or need for can openers, etc.) in mind.

This could produce an interesting discussion on design and mathematics, etc.

11 Approximating irregular spaces: integration and differential equations

Essential understandings

Calculus describes rates of change between two variables and the accumulation of limiting areas. Understanding these rates of change allows us to model, interpret and analyze real-world problems and situations. Calculus helps us understand the behavior of functions and allows us to interpret the features of their graphs.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Students will understand the relationship between the integral and area and interpret the meaning of this in context.
- Numerical integration can be used to approximate areas in the physical world.
- The area under a function on a graph has a meaning and has applications in space and time.
- Kinematics allows us to describe the motion and direction of objects in closed systems in terms of displacement, velocity and acceleration.
- Many physical phenomena can be modelled using differential equations and analytic and numeric methods can be used to calculate optimum quantities.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
The area under a curve in a given interval can be found by finding lower and upper rectangle sums. When the number of rectangles tends to infinity, the lower and upper sums tend to the same number which is the area under the curve. This number is definite integral of the curve in the given interval.	Investigation 1
Areas underneath the x-axis will be represented as negative integrals and this must be taken into consideration when evaluating areas for a definite integral.	Investigation 2
The area underneath a curve may be approximated using the sum of any number of trapeziums and the limiting sum leads to more accuracy.	Investigation 3
The process of finding the indefinite integral of a function leads to a family of functions that differ by a constant.	Investigation 4
The anti-derivative represents the inverse of the derivative so can be evaluated using the inverse of all differentiation rules.	Investigation 5
The area under a velocity–time graph represents the total distance travelled by the particle, found by taking the limiting sum of all the displacements (velocity multiplied by time).	Investigation 6
The complex exponential function with base e may be expressed as a trigonometric expression with a real and imaginary part.	Investigation 7

Syllabus sections covered in this chapter:

- SL5.5*
- SL5.8
- AHL5.11
- AHL5.12
- AHL5.13
- AHL5.14
- AHL5.15

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 473	Page 493: Example 10 Page 505: Example 17 Page 512: Example 21	Page 479: Example 2 Page 479: Example 3 Page 480: Example 4 Page 496: Example 12 Page 499: Example 14 Page 499: Example 15 Page 500: Example 16 Page 505: Example 17 Page 506: Example 18 Page 512: Example 21 Page 522: Example 25	Pages 485, 498, 513, 517, 523

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 524	Page 526	Page 529

11.1 Finding approximate areas for irregular regions

Investigation 1

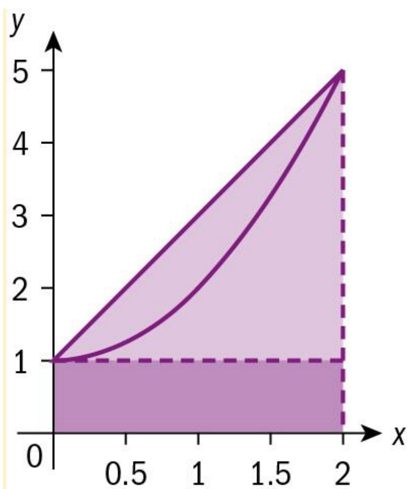
Conceptual understanding:

The area under a curve in a given interval can be found by finding lower and upper rectangle sums. When the number of rectangles tends to infinity, the lower and upper sums tend to the same number which is the area under the curve. This number is definite integral of the curve in the given interval.

Additional notes

It is suggested that teachers encourage students to use Geogebra to see how the lower and the upper sums tend to the same value as the number of rectangles tends to infinity as well as to see how the rectangles start covering the actual area. However, the first steps of the investigation should be done by hand to show them how these values are found.

1 A possible estimation could be using a trapezium or a rectangle and a triangle.



Using the trapezium area formula:

$b_1 = 1$, $b_2 = 5$ and $h = 2$.

$A \approx \frac{2}{2}(1+5) = 6u^2$

Splitting the area into a rectangle and a triangle:

Area of rectangle = 2 × 1 = 2

Area of triangle = $\frac{2 \times 4}{2}$ =4.

Area total = 6 therefore

$A \approx 6u^2$

Discuss the idea of overestimation and underestimation with the students. In this case we are overestimating because the area we want to find is clearly smaller than the area that we are calculating.

You can ask *whether this value can be improved and how?* There may be students that start thinking on the trapezium rule idea which would be good for the next investigation.

2 **a** The length of the interval is equal to 2 units and all the rectangles have the same width.

This means that the width will be found by dividing the length of the interval by the number of rectangles.

Width of rectangles = $\frac{2-0}{4}$ = 0.5.

b The base of each rectangle lies between two values of *x* and is found by replacing the smaller value of *x* in the formula of the function.

c

	x_i	Height = $f(x_i)$
R_1	0	$f(0)=1$
R_2	0.5	$f(0.5)=1.25$
R_3	1	$f(1)=2$
R_4	1.5	$f(1.5)=3.25$

d

	x_i	Height = $f(x_i)$	Area of the rectangle = 0.5 × height
R_1	0	$f(0)=1$	$1\times0.5=0.5$
R_2	0.5	$f(0.5)=1.25$	$1.25\times0.5=0.625$
R_3	1	$f(1)=2$	$2\times0.5=1$
R_4	1.5	$f(1.5)=3.25$	$3.25\times0.5=1.625$

Sum of the areas of rectangles = 0.5 + 0.625 +1 +1.625 = 3.75

e This value is an underestimation of the actual area. It is clear from the graph that there is part of the actual area that this area is not covering.

3 **a** The widths are all equal to 0.5.

b The heights of the rectangles are as follows:

	x_i	Height = $f(x_i)$
R_1	0.5	$f(0.5)=1.25$
R_2	1	$f(1)=2$
R_3	1.5	$f(1.5)=3.25$
R_4	2	$f(2)=5$

c Again a table is useful to organize information.

	x_i	Height = $f(x_i)$	Area of the rectangle = 0.5 × height
R_1	0.5	$f(0.5)=1.25$	$1.25\times0.5=0.625$
R_2	1	$f(1)=2$	$2\times0.5=1$
R_3	1.5	$f(1.5)=3.25$	$3.25\times0.5=1.625$

R_4	2	$f(2) = 5$	$5 \times 0.5 = 2.5$

Sum of the areas of rectangles = 0.625 + 1+1.625 + 2.5 = 5.75

d This is an overestimation of the actual area. We are considering in the calculation more area than what the actual region covers.

4 a The more rectangles we use, the closer our approximations are to the actual area.

b Lower bound with 6 rectangles:

Rectangle	Width	Height	Area
R_1	$\frac{1}{3}$	$f(0) = 1$	$\frac{1}{3}$
R_2	$\frac{1}{3}$	$f(\frac{1}{3}) = \frac{10}{9}$	$\frac{1}{3} \times \frac{10}{9} = \frac{10}{27}$
R_3	$\frac{1}{3}$	$f(\frac{2}{3}) = \frac{13}{9}$	$\frac{1}{3} \times \frac{13}{9} = \frac{13}{27}$
R_4	$\frac{1}{3}$	$f(1) = 2$	$\frac{1}{3} \times 2 = \frac{2}{3}$
R_5	$\frac{1}{3}$	$f(\frac{4}{3}) = \frac{25}{9}$	$\frac{1}{3} \times \frac{25}{9} = \frac{25}{27}$
R_6	$\frac{1}{3}$	$f(\frac{5}{3}) = \frac{34}{9}$	$\frac{1}{3} \times \frac{34}{9} = \frac{34}{27}$
			$L_S = 4.03704$

Upper bound with 6 rectangles:

Rectangle	Width	Height	Area
R_1	$\frac{1}{3}$	$f(\frac{1}{3}) = \frac{10}{9}$	$\frac{1}{3} \times \frac{10}{9} = \frac{10}{27}$
R_2	$\frac{1}{3}$	$f(\frac{2}{3}) = \frac{13}{9}$	$\frac{1}{3} \times \frac{13}{9} = \frac{13}{27}$
R_3	$\frac{1}{3}$	$f(1) = 2$	$\frac{1}{3} \times 2 = \frac{2}{3}$
R_4	$\frac{1}{3}$	$f(\frac{4}{3}) = \frac{25}{9}$	$\frac{1}{3} \times \frac{25}{9} = \frac{25}{27}$
R_5	$\frac{1}{3}$	$f(\frac{5}{3}) = \frac{34}{9}$	$\frac{1}{3} \times \frac{34}{9} = \frac{34}{27}$
R_6	$\frac{1}{3}$	$f(2) = 5$	$\frac{1}{3} \times 5 = \frac{5}{3}$
			$U_S = 5.37037$

c The lower bound has increased and the upper bound has decreased. They are now closer than when the approximation was made with 4 rectangles.

The approximation of the area can be improved if we increase the number of rectangles.

d 4.03704 < A < 5.37037

5 **a** As n increases the values of L_S increase as well. They increase every time with a lower rate.

b As n increases the values of U_S decrease. They decrease every time with a lower rate.

c When n tends to infinity it can be seen that the values of L_S increase and tend to 4.66...

The values of L_S increase in a way that the difference between them is every time smaller and smaller.

When n tends to infinity it can be seen that the values of U_S decrease and tend to 4.66...

The values of U_S decrease in a way that the difference between them is every time smaller and smaller.

Both the upper and lower sums approach the same number as n tends to infinity. This number is the value of A, which is in between L_S and U_S.

6 The lower limit is $x = 0$, the upper limit is $x = 2$. The function is positive between $x = 0$ and $x = 2$.

7 **Conceptual:** How do areas under curves within a given interval relate to the definite integral and to lower and upper rectangle sums on the same interval?

Answer: The area under a curve in a given interval can be found by finding lower and upper rectangle sums. When the number of rectangles tends to infinity, the lower and upper sums tend to the same number which is the area under the curve. This number is definite integral of the curve in the given interval.

TOK

Where does mathematics come from?

Galileo said that the universe is a grand book written in the language of mathematics

Does it start in our brains or is it part of the universe?

Mathematics may not be the language of the universe, but rather the language/logical system the brain uses to analyse and respond to the universe. Ask students what they think about this and to consider where the real foundations of mathematics originate.

You might want to use facts such as The Higgs Boson was predicted with the same tool as the planet Neptune and the radio wave: with mathematics, which might mean that our universe isn't just described by mathematics, but that it is mathematics in the sense that we're all parts of a giant mathematical object.

Max Tegmark and Brian Butterworth provide some interesting insights and contrasting opinions on YouTube.

Reflect: What is a definite integral?

Answer: A definite integral represents the area under a curve for positive functions and the only number that lies between all lower and upper rectangle sums.

Investigation 2

Conceptual understanding:

Areas underneath the x-axis will be represented as negative integrals and this must be taken into consideration when evaluating areas for a definite integral.

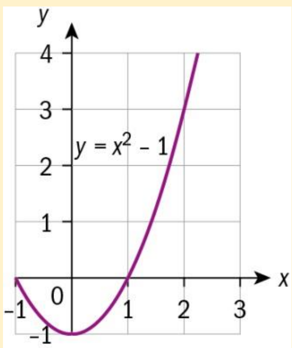
1 **a** $-0.666\ldots$

b $1.333\ldots$

c $0.666\ldots$

2 $C = A + B$

3 **a**

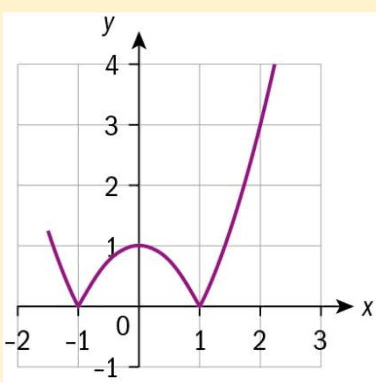


b Because the area between $x = 0$ and $x = 1$ is given a negative value

c **Factual:** What do you notice about the definite integral when a function is below the x-axis?

Answer: When the area is completely below the x-axis, the definite integral is negative.

4 **a**



b Calculate $\int_0^2 \left|x^2 - 1\right| dx$

5 **Conceptual:** When evaluating areas bounded by curves what must you consider when evaluating the definite integral?

Answer: Areas underneath the x-axis will be represented as negative integrals and this must be taken into consideration when evaluating areas for a definite integral.

Investigation 3

Conceptual understanding:

The area underneath a curve may be approximated using the sum of any number of trapeziums and the limiting sum leads to more accuracy.

3 **a** 21.501

b 1

c 12, 6, 4, 3, 2.4, 2

d 22.4

e $\left|\dfrac{21.501-22.4}{21.501}\right|\times 100 = 4.18\%$

f $\frac{5}{10} = 0.5$

g 12, 8, 6, 4.8, 4, 3.43, 3, 2.67, 2.4, 2.18, 2

h $\left|\dfrac{21.501-21.74}{21.501}\right|\times 100 = 1.11\%$

i The limit will be the area under the curve ≈ 21.501.

4 **Conceptual:** How can we estimate the area underneath a curve?

5 **Conceptual:** How do we make this approximation more accurate?

Answers to 4 and 5: The area underneath a curve may be approximated using the sum of any number of trapeziums and the limiting sum leads to more accuracy.

Developing inquiry skills

In the opening problem for this chapter you looked at how to estimate the area of San Cristobal.

Use the trapezoidal rule to work out an estimate of the area using the points given. Compare your answer with the officially given figure of 558 km^2.

Answer:

$$\frac{1}{2}\times 9\left(15+37+2\left(22+27.5+37+40\right)\right)=1372.5$$

$$\frac{1}{2}\times 9\left(15+37+2\left(11+12.5+16+26\right)\right)=823.5$$

$$1372.5-823.5=549\ \text{km}^2$$

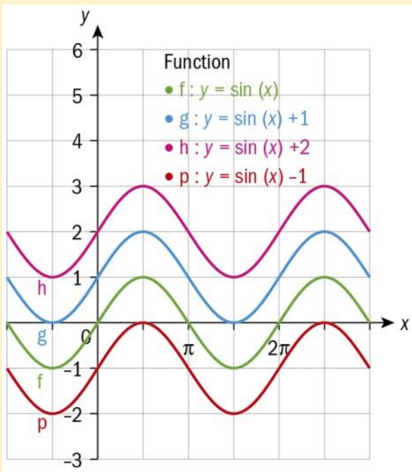
11.2 Indefinite integrals and techniques of integration

Investigation 4

Conceptual understanding:

The process of finding the indefinite integral of a function leads to a family of functions that differ by a constant.

1



2 The gradients at $x = \frac{\pi}{2}$ are all equal to 0

3 The gradients at x= 0 are all equal to 1.

4 The gradients at x are all $\cos x$. Because the derivative of a constant is zero

5 For example $y = \sin x + 27.5$

6 The general formula is $y = \sin x + c$, where c is any constant.

7 **Conceptual:** What does finding the indefinite integral lead to?

Answer: The process of finding the indefinite integral of a function leads to a family of functions that differ by a constant.

TOK

Is imagination more important than knowledge?

"Imagination is the beginning of creation. You imagine what you desire, you will what you imagine and at last you create what you will." George Bernard Shaw

Albert Einstein thought so. He said: "I'm enough of an artist to draw freely on my imagination, which I think is more important than knowledge.

Knowledge is limited. Imagination encircles the world."

When you see your students share the knowledge they have learned from you, don't you feel proud?

Now, when you see their imagination use that knowledge and take a step further, that's amazing.

Have students write about which they think is the more important, and why.

Reflect: What is an indefinite integral?

Answer: An indefinite integral defines a family of antiderivatives.

Investigation 5

Conceptual understanding:

The anti-derivative represents the inverse of the derivative so can be evaluated using the inverse of all differentiation rules.

1 **a** $\frac{1}{2}x^2+c$

b $\frac{1}{3}x^3+c$

c $\frac{1}{4}x^4+c$

2 Hopefully students will pick up the obvious patter and conjecture $\frac{1}{n+1}x^{n+1}+c$. Some may already spot the problem that occurs when *x*=-1, if so they can begin considering what the solution might be by thinking about what might differentiate to give $\frac{1}{x}$.

3 **a** $\frac{1}{6}x^6+c$

b $\frac{2}{3}x^{\frac{3}{2}}+c$

c $-x^{-1}+c$ or $-\frac{1}{x}+c$

4 Based on their knowledge of the rules for differentiation they should get the answer $\frac{1}{n+1}ax^{n+1}+c$ or $\frac{a}{n+1}x^{n+1}+c$

5 Using the rule above the anti-derivative is $\frac{12}{6}x^6+\frac{3}{3}x^3+\frac{4}{2}x^2+7x+c$ $=2x^6+x^3+2x^2+7x+c$. When differentiated this gives the original function $12x^5+3x^2+4x+7$.

6 **a** $\frac{1}{0}x^0+c$

b This is undefined as you cannot divide by 0

c In chapter 10 students learned that for $x > 0$ the derivative of $\ln x$ is $\frac{1}{x}$ and hence the antiderivative of $\frac{1}{x}$ is $\ln x + c$

d We know that when $x > 0$ the antiderivative is $\ln x$ which is the same as $\ln|x|$, when $x < 0$ then the derivative of $\ln|x|$ will be the derivative of $\ln(-x)$ (which is well defined as $x < 0$) which will be $-\frac{1}{-x}$ (by the chain rule) which is equal to $\frac{1}{x}$. So for all values of $x \neq 0$ the derivative of $\ln|x|$ is $\frac{1}{x}$ and hence the antiderivative of $\frac{1}{x}$ is $\ln|x|$.

7 The antiderivatives can be found either using the work earlier in the previous investigation, or by using the results from chapter 10.

a $-\cos x + c$

b $\sin x + c$

c $\tan x + c$

d $e^x + c$

8 **a** **i** $f'(x) = 4\cos 4x$

ii $f'(x) = -5\sin 5x$

iii $f'(x) = 7e^{7x}$

b **i** $\frac{1}{4}\sin 4x + c$

ii $-\frac{1}{5}\cos 5x + c$

iii $\frac{1}{7}e^{7x} + c$

c **i** $\frac{1}{a}\sin ax + c$

ii $-\frac{1}{a}\cos ax + c$

iii $\frac{1}{a}e^{ax} + c$

9 **Factual:** What are the anti-derivative rules?

Answer: The following rules are given in the formula book:

$$\int x\mathrm{d}x = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$\int \frac{1}{x}\mathrm{d}x = \ln|x| + C$$

$$\int \sin x\mathrm{d}x = -\cos x + C$$

$$\int \cos x\mathrm{d}x = \sin x + C$$

$$\int \frac{1}{\cos^2 x} = \tan x + C$$

$$\int e^x dx = e^x + C$$

In addition it would be useful for students to learn the ones conjectured above, namely

$$\int \cos ax\mathrm{d}x = \frac{1}{a}\sin ax + c \qquad \int \sin ax\mathrm{d}x = -\frac{1}{a}\cos ax + c \qquad \int \mathrm{e}^{ax}\mathrm{d}x = \frac{1}{a}\mathrm{e}^{ax} + c$$

10 **Conceptual:** How can you evaluate anti-derivatives given what you already know about derivatives?

Answer: The anti-derivative represents the inverse of the derivative so can be evaluated using the inverse of all differentiation rules.

TOK

We are trying to find a method to evaluate the area under a curve.

"The main reason knowledge is produced is to solve problems." To what extent do you agree with this statement?

A useful starting point for many TOK questions is to identify the key themes and words in the knowledge question and look for examples, claims and counterclaims.

What do we mean by a problem?

What instances do you know of where mathematics has been created to solve a problem?

Can you have knowledge that does not solve problems?

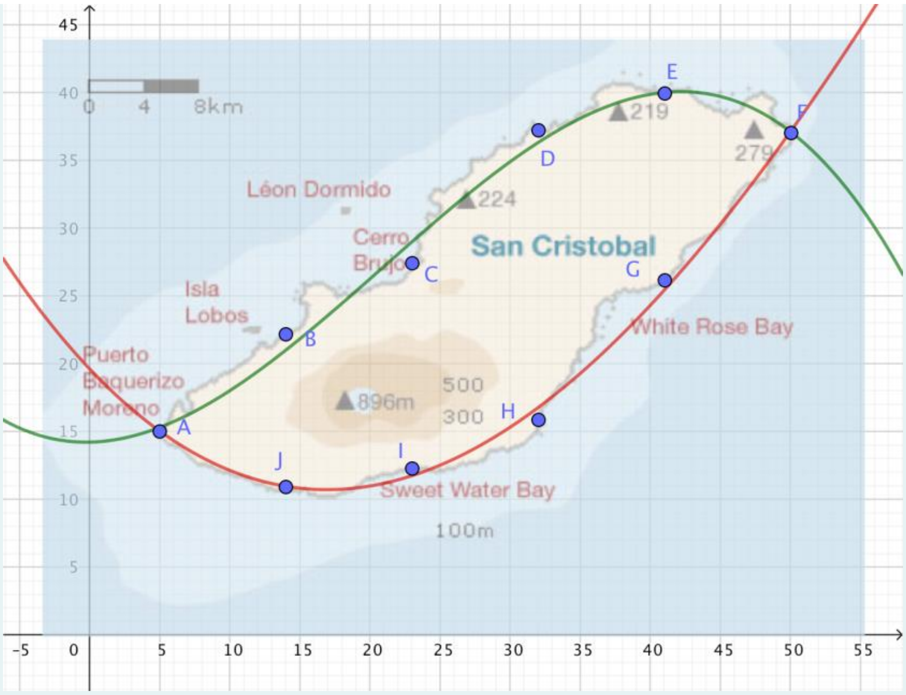
11.3 Applications of integration

Developing inquiry skills

Look again at the opening problem. To work out the area, we will now fit two curves through the points found previously: one curve through A, B, C, D, E and F and one through A, J, I, H, G and F.

Use cubic regression to find an equation for each curve and hence, an estimate for the area of San Cristobal.

Answer:



The two curves are $y = -0.000686x^3 + 0.0434x^2 + 0.00279x + 14.3$ and

$y = -0.000114x^3 + 0.00335x^2 - 1.03x + 19.3$

The area is approximately $1379 - 808 = 571$ km^2.

Students could judge from the picture above or from their own functions how accurate this method was and whether it is an under or over estimate. They could also be asked whether they felt the estimate from the trapezium rule (549 km^2) or the value found from regression is more accurate.

The actual area of the island is given as 558km^2, so both estimate are accurate to within 2.5%.

TOK

Consider $f(x) = \frac{1}{x}, 1 \le x \le \infty$

An infinite area sweeps out a finite volume.

How does this compare to our intuition?

What does this tell us about mathematical knowledge?

Gabriel's horn is an interesting paradox. You have a hollow object which has infinite length. It can be filled with paint, but if you try to cover the surface with that paint, there won't be enough. How does that make sense? What does your intuition tell you?

Some say that "When Gabriel blows the horn we all go to paradise". The archangel Gabriel is shared by Judaism, Islam and Christianity.

TOK

You have been using radians to measure angles instead of degrees in recent chapters.

Why has this change been necessary? What are its advantages?

Maybe there is a better measure than radians in terms of pi.

The number tau is not as well-known as pi, but became more popular when Michael Hartl wrote "The Tau Manifesto" and it continues to gain support.

Research tau and decide if you think it is superior to pi.

This shows you that as certain conventions come about in mathematics it is very hard to go back if the choice was not ideal. Mathematics would be much easier if we used tau, but we don't and to change would take a massive effort. Some also argue that there is no need for change.

It shows that we can have opinions in mathematics. There is a debate about what is best. This is normally seen in subjects like Human Sciences, Art, Ethics…. And is perhaps not expected in Mathematics.

What does this argument tell us about the nature of mathematics?

How is this different or similar to other Areas of Knowledge?

TOK

Does the inclusion of kinematics as core mathematics reflect a particular cultural heritage?

Students often ask "Why are we doing this?" with the pure mathematics sections and kinematics is a good real-life application, but should kinematics be in science or mathematics?

You might want to ask – Who decides what is mathematics?

Using a cultural context to integrate science and mathematics teaching and learning using students' daily life experiences, cultural heritage can promote awareness, sensitivity, appreciation, respect, and pride in community and culture.

Investigation 6

Conceptual understanding:

The area under a velocity–time graph represents the total distance travelled by the particle, found by taking the limiting sum of all the displacements (velocity multiplied by time).

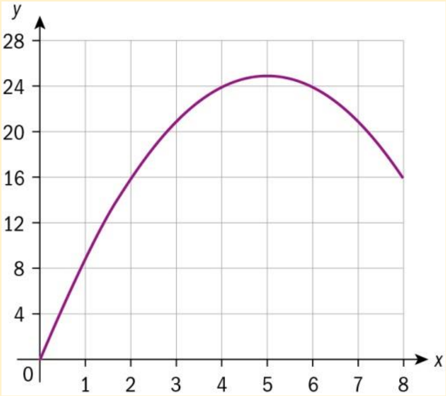
1 $v = 0$ and because velocity goes from positive to negative it must be a maximum.

2 **a** $s = 10t - t^2 + c$

When $t = 0, s = 0, c = 0$

Hence $s = 10t - t^2$.

b



3 **a** $s = \mathbf{10} \times \mathbf{5} - 5^2 = \mathbf{25}$ m

b $s = 10 \times 8 - 8^2 = 16$

4 25 + 9 = 34 m

5 Using formula for area of a triangle, area is $\frac{1}{2} \times 10 \times 5 + \frac{1}{2} \times 3 \times 6 = 34$.

6 **Conceptual:** How do you find the distance travelled by a particle?

Answer: The area under a velocity–time graph represents the total distance travelled by the particle, found by taking the limiting sum of all the displacements (velocity multiplied by time).

7 Because the velocity and hence the integral is negative for part of the time interval.

TOK

Why do we study mathematics?

What's the point?

Can we do without it?

At this level of complex mathematics, some students may well be asking this question!

Ask students what words come to mind when they think about the word mathematics.

Ask them where they see mathematics in everyday life.

Galileo said that we find mathematics everywhere in nature. Ask students to illustrate some examples.

Ask students where they benefit from mathematics.

You could create a wall or board or website display of the answers.

Maybe you could ask them to describe the scenario of a world without mathematics, highlighting the positives and negatives

11.4 Differential equations

Investigation 7

Conceptual understanding:

The complex exponential function with base e may be expressed as a trigonometric expression with a real and imaginary part.

1 The solution is performed using the technique of separating the variables.

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \mathrm{i}x \Rightarrow \int \frac{1}{x}\,\mathrm{d}x = \int \mathrm{i}\,\mathrm{d}\theta$$

$$\Rightarrow \ln x = \mathrm{i}\theta + c \Rightarrow x = A\mathrm{e}^{\mathrm{i}\theta} \text{ but } x = 1 \text{ when } \theta = 0 \text{ so } A=1 \text{ and } x = \mathrm{e}^{\mathrm{i}\theta}$$

2 When $\theta = 0$ $x = \cos 0 + \mathrm{i}\sin 0 = 1$

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\sin\theta + \mathrm{i}\cos\theta = \mathrm{i}\left(\mathrm{i}\sin\theta + \cos\theta\right) = \mathrm{i}x$$

3 **Conceptual:** What can you conclude about $\mathrm{e}^{\mathrm{i}\theta}$?

Answer: The complex exponential function with base e may be expressed as a trigonometric expression with a real and imaginary part.

Investigation 8

Investigation 8 takes the student through an explanation. No answers are required.

Developing inquiry skills

Write down any further inquiry questions you could ask and investigate how you could find the areas of irregular shapes and curved shapes.

Answer: Students' own answers.

In the footsteps of Archimedes

Approaches to Learning/learner profile: Research, Critical Thinking

Exploration Criteria: Mathematical communication (B), Personal engagement (C), Use of mathematics (E)

IB Topic: Integration, Proof, Coordinate Geometry

Introduction

This task looks at a topic in the vast History of Mathematics. Many students in fact write their explorations on the history of mathematics. Clearly this demonstrates that students are interested. This is fine, but a mere historical account copied or summarized from the internet is unlikely to score well on most criteria, especially if there is little or no mathematics actually used. There is often very little or no opportunity for reflection or engagement in this type of exploration. Having said that, it is still worthwhile including aspects of mathematical history in class to give the subject or a particular topic some context. It will also provide students with examples that they may incorporate into their TOK lessons and assessments.

This particular task looks at Archimedes' method for finding the area of a parabolic segment and is a remarkable result. This is interesting as he worked on this almost 2000 years before Newton and Leibniz formalized differential and integral calculus. Archimedes, in his time and in his own way, already had a good grasp of the basics of calculus but despite this he is not really part of the debate about the Father of Calculus!

The task is set up so that there are opportunities to discover aspects of his proof and consider ways that Personal engagement (Criterion C), Mathematical communication (Criterion B) and Use of mathematics (Criterion E) can be demonstrated. There are also clear links between different areas of mathematics covered in the syllabus which, depending on the approach to the discussion, can also be credited in Criterion C and Criterion D.

The area of a parabolic segment

This task looks at Archimedes’ work on finding the area of a parabolic segment.

To set the scene, you could ask students what they know about Archimedes.

Archimedes was a famous Greek mathematician working 2300 years ago. He has many theorems and discoveries credited to him and is generally regarded as one of the greatest mathematicians of all time.

You could also talk about the history of calculus, and who is credited for any breakthroughs.

Newton and Leibnitz are credited with the invention of calculus and the methods used in this chapter (although there is still some controversy over exactly who developed it first between these two), but the most remarkable thing about Archimedes' work is that it predates their work by around 2000 years.

Archimedes would not have used the coordinate system as we know it today since it was not invented until the 17th century by Descartes, but his method would have been similar.

The x-value is 0.5 as this is halfway between −2 and 3.

The corresponding y-value is $0.5^2 = 0.25$.

So $C(0.5, 0.25)$.

Students can verify the result in this case by calculating the area of the triangle and the area of the shaded area.

In the given example, the line $y = x + 6$ intersects the parabola at A(−2, 4) and B(3, 9) and the other vertex of the triangle on $y = x^2$ is at C(0.5, 0.25).

Methods for calculating the area of the triangle include:

- Work out the length of AC and the perpendicular height from AC extended to B and use the formula Area = 1/2bh.
- Use the formula for calculating the area which uses the three vertices of the triangle:

$$\text{area} = \left|\frac{Ax(By - Cy) + Bx(Cy - Ay) + Cx(Ay - By)}{2}\right|$$

where A_x and A_y are the x- and y-coordinates of the point A, etc.

- Use the bound box method. This is where a box is drawn around the triangle and the area of around the edge of the required triangle area is calculated and subtracted from the area of the box.
- Find the length of the three sides and then use Heron’s formula.

Whichever method used, area $\Delta ABC = 15.625 = 15\frac{5}{8}$ unit^2

Students can use integration to calculate the required area between the two curves. The upper curve is the line $y_2 = x + 6$ and the lower curve is $y_1 = x^2$. The limits of integration are $x = -2$ and $x = 3$.

$$\int_{-2}^{3} \left[(x+6)\right] - x^2 \, dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3}$$

$$= 20\frac{5}{6}$$

So the area between the two curves is $20\frac{5}{6}$ unit^2.

If needed, assist students with the above integration.

So according to Archimedes, the area of the parabolic segment will be:

Area of segment $= \frac{4}{3} \times 15\frac{5}{8} = 20\frac{5}{6}$ unit^2 as required.

You could discuss how Archimedes proved his result:

The way Archimedes achieved this result was by **inscribing** successively smaller polygons (triangles in this case) until the area was filled. You can calculate the sum of the area of these (infinitely many) triangles and hence the area of the curved shape.

D(−0.75, 0.5625)

Students could use one of the above methods to calculate the area of triangle ACD.

Area of triangle ACD = 1.953125 unit^2

E(1.75, 3.0625)

Area of triangle BCE = 1.953125 unit^2

Ratio is $\frac{1.953125}{15.625} = \frac{1}{8}$.

The area of each new triangle is $\frac{1}{8}$ the area of the big triangle. (The sum of the two triangles is $\frac{1}{4}$ the area of the big triangle)

This can be **proved** generally to always be the case, because the new triangles have $\frac{1}{2}$ the width of the original triangle and $\frac{1}{4}$ of the height.

Note that with the 7 triangles, as before, the new triangles will be $\frac{1}{8}$ the area of the previous triangles. In fact, it can be shown that this is true of any triangles inscribed in such a way within the parabola.

Again, students could be encouraged to work on this or research reasons why.

You could improve the approximation by using more triangles.

It is possible to speed up calculations by constructing a spreadsheet. This should allow you to easily add increasingly more triangles. However, this will not be a proof.

Students could be encouraged to do this or at least note that this is a possibility in this sort of task and could be a good approach if it is an exploration to demonstrate personal engagement.

Generalize the problem

Archimedes originally used a geometric proof. The method shown here, however, uses mathematics covered earlier in the course – the sum of an infinite geometric sequence.

Total area of the next two triangles = $2\left(\frac{X}{8}\right)=\frac{X}{4}$.

Total area of the next four triangles = $4\left(\frac{X}{64}\right)=\frac{X}{16}$

Total area of the next eight triangles = $8\left(\frac{X}{512}\right)=\frac{X}{64}$

$$X+2\left(\frac{X}{8}\right)+4\left(\frac{X}{64}\right)+8\left(\frac{X}{512}\right)+\ldots$$

$$X=\frac{X}{4}+\frac{X}{16}+\frac{X}{64}+\ldots$$

This is a geometric series, with common ratio $r=\frac{1}{4}$ and first term $U_1 = X$.

The sum of this series is given by: Sum = $\frac{U_1}{1-r}=\frac{X}{1-\frac{1}{4}}=\frac{X}{\frac{3}{4}}=\frac{4X}{3}$

The sum of the areas of all the triangles gives the area of the parabolic segment and this is $\frac{4X}{3}$, which is $\frac{4}{3}$ the area of the original triangle, just as Archimedes claimed!

Archimedes' thinking behind this solution is very similar to the ideas behind the development of calculus as seen in this chapter.

Extension

Studying the history of mathematics helps students develop a deeper understanding of the mathematics they have already studied by seeing how it was developed over time and, importantly, in various places across the world. It also encourages creative and flexible thinking by allowing students to see historical evidence that there are different and perfectly valid ways to view concepts and to carry out computations.

There is a clear link to TOK too.

Consider this from the knowledge framework in the IB TOK guide for mathematics:

Historical development

How has our understanding and perception of mathematics changed over time? How has the role of mathematics within society developed? To what extent has the nature of mathematics (for example, the different forms of mathematics) changed? What relationship does today's mathematical understanding have with that of the past? (to paraphrase Newton, does it "stand on the shoulders of giants"?)

Here are a couple of interesting websites to explore regarding the history of mathematics:

https://www.storyofmathematics.com/

http://www-groups.dcs.st-and.ac.uk/~history/

Again, it is important to counsel students on writing explorations purely on the history of a mathematical topic unless: Use of mathematics (Criterion E) must be clearly demonstrated and Personal engagement (Criterion C) should be considered beyond just "this is interesting" or "this is important" and Reflection (Criterion D) beyond just "this is hard".

12 Modelling motion and change in two and three dimensions

Essential understandings

Calculus describes rates of change between two variables and the accumulation of limiting areas. Understanding these rates of change allows us to model, interpret and analyze real-world problems and situations. Calculus helps us understand the behavior of functions and allows us to interpret the features of their graphs.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Many physical phenomena can be modelled using differential equations and analytic and numeric methods can be used to calculate optimum quantities.
- Phase portraits enable us to visualize the behavior of dynamic systems.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	**Investigation**
Scalar and vector products can be used to find the component of a vector ***a*** acting parallel and perpendicular to a vector ***b***.	Investigation 1
Vectors can be used to describe projectile motion by specifying the horizontal and vertical components and integrating the acceleration equation.	Investigation 2
Vectors can describe angular velocity in circular motion by differentiating the displacement equation and acceleration can be found by differentiating the angular velocity equation.	Investigation 3
Eigenvalues and eigenvectors can be used to “decouple” a system to give equations in a single variable.	Investigation 4
A phase portrait displays geometrical representations of future paths, in other words how different variables change over time with different initial values and they can depict the solutions of differential equations.	Investigation 5
A matrix for a coupled system with imaginary eigenvalues indicates periodic motion, while complex eigenvalues display motion forming a spiral.	Investigation 6
A predator–prey model normally displays a periodic rise or fall of each population and can be developed from a system of coupled differential equations which can be solved using numerical methods.	Investigation 7

Syllabus sections covered in this chapter:

- AHL3.12
- AHL3.13
- AHL5.16
- AHL5.17
- AHL5.18

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 533	Page 538: Example 2 Page 545: Example 4 Page 549: Example 5 Page 556: Example 6	Page 556: Example 6	539, 546, 555, 561

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 614	Page 616	N/A

Developing inquiry skills

An aircraft needs to deliver a supply package to a polar research station. The package will be dropped from the aircraft from a height of 150 m. If the aircraft is flying at 180 km h^{-1} how far from the research station would the package need to be released if air resistance can be ignored?

- Make a guess at a possible answer to the question above.
- How would your answer change if the aircraft still had the same speed at the same height but was ascending at an angle of 30°?
- If air resistance cannot be ignored would the package need to be released further from or closer to theresearch station?

Answer: Students who have studied some Physics could make a good guess at the solution to this question. They need to work out an approximate time for the parcel to fall 150 m and then decide how far horizontally from the research station this would be. One issue they will need to consider is the units that are given in the question.

In the second question the change of angle means that the parcel now also has a vertical velocity, so will take longer to reach the ground. But is this compensated for by the fact that the horizontal velocity of the plane is also reduced?

In the case of air resistance, the parcel will take longer to reach the ground so that means it could be released further away from the research station, but its horizontal velocity will also be reduced, so will this be the more significant factor?

These questions will be answered at various point in the chapter.

12.1 Vector quantities

Investigation 1

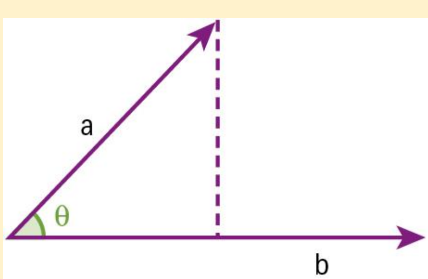
Conceptual understanding:

Scalar and vector products can be used to find the component of a vector **a** acting parallel and perpendicular to a vector **b**.

Additional notes

The purpose of the investigation is for the students to derive the formula which gives the component of one vector in the same direction as another (and perpendicular to that vector).

1 and **2** The students may need guiding to the fact that a coordinate system is not needed in this question and ***a*** and ***b*** can have any direction.



From the diagram it is clear by right-angled trigonometry, that the two components are $|\boldsymbol{a}|\cos\theta$ and $|\boldsymbol{a}|\sin\theta$.

3 $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ hence component in the direction of ***b*** is $\frac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{b}}$.

4 Similarly $|\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$ hence component perpendicular to ***b*** within the plane defined by the two vectors is $\frac{|\boldsymbol{a}\times\boldsymbol{b}|}{\boldsymbol{b}}$

5 **a**

$$\frac{\begin{pmatrix}1\\2\\1\end{pmatrix}\bullet\begin{pmatrix}-3\\4\\0\end{pmatrix}}{\sqrt{(-3)^2+4^2}} = \frac{5}{5} = 1$$

b

$$\frac{\left|\begin{pmatrix}1\\2\\1\end{pmatrix}\times\begin{pmatrix}-3\\4\\0\end{pmatrix}\right|}{\sqrt{(-3)^2+4^2}} = \frac{\left|\begin{pmatrix}-4\\-3\\10\end{pmatrix}\right|}{5}$$

$$= \frac{\sqrt{125}}{5} = 2.24$$

6 **Conceptual:** How do you find the component of a vector ***a*** acting

i parallel to a vector ***b*** **ii** perpendicular to a vector ***b***

Answer: Scalar and vector products can be used to find the component of a vector ***a*** acting parallel and perpendicular to a vector ***b.***

The particular equations used are given in the formula book.

12.2 Motion with variable velocity

TOK

Do you think that one form of symbolic representation is preferable to another?

The symbols used internationally for mathematics differ across countries and teachers.

Research and display some such as $y =$ and $f(x)$, the different ways of showing the domain, some using a comma as a decimal point, etc.

Which ones should you use? Why? Should we be exposed to all of the possibilities?

TOK

Why might it be argued that vector equations are superior to Cartesian equations?

A good class discussion as students might be more confident with previously encountered work.

Investigation 2

Conceptual understanding:

Vectors can be used to describe projectile motion by specifying the horizontal and vertical components and integrating the acceleration equation.

Additional notes

This investigation leads to some of the standard results of projectile motion. It is not expected that students learn these formulae. All questions set on projectiles would require student to derive any results needed from the particular information given in the question.

The exercise though will give students practice at working with general formulae and give them some insight into the applications of the principles being developed in the chapter.

1 There is no horizontal component for acceleration, and because it acts down the vertical component is shown as $-g$ $\begin{pmatrix} 0 \\ -g \end{pmatrix}$.

2 Drawing a diagram and using right-angled trigonometry should enable students to find this vector, $\begin{pmatrix} u\cos\alpha \\ u\sin\alpha \end{pmatrix}$. The technique is particular case of finding the component of a vector in a given direction which was considered in the first investigation.

3 The velocity equation is obtained by integrating the acceleration equation and using the initial velocity to find the constants of integration (u_1 and u_2).

$$\boldsymbol{v} = \begin{pmatrix} \int 0\mathrm{d}t \\ \int -g\mathrm{d}t \end{pmatrix} = \begin{pmatrix} u_1 \\ -gt + u_2 \end{pmatrix}$$

When t=0 the velocity is $\begin{pmatrix} u\cos\alpha \\ u\sin\alpha \end{pmatrix}$.

Hence $\boldsymbol{v} = \begin{pmatrix} u\cos\alpha \\ u\sin\alpha \end{pmatrix}$.

4 The displacement equation is found by integrating the velocity equation and substituting the initial values, to find the constants of integration.

$$r = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (u\cos\alpha)t + c_1 \\ (u\sin\alpha)t - \frac{1}{2}gt^2 + c_2 \end{pmatrix}$$

5 The particle will reach its maximum height when its vertical velocity is equal to zero. Hence $u\sin\alpha - gt = 0$ and so $t = u\frac{\sin\alpha}{g}$.

6 The height of the particle is given by the y (vertical) component of the displacement vector.

The particle is projected from (0, 0) so both c_1 and c_2 are equal to zero.

$$y = (u\sin\alpha)t - \frac{1}{2}gt^2 \quad \text{so when } t = \frac{u\sin\alpha}{g}$$

$$y = (u\sin\alpha)\frac{u\sin\alpha}{g} - \frac{1}{2}g\left(\frac{u\sin\alpha}{g}\right)^2 = \frac{u^2\sin^2\alpha}{g} - \frac{gu^2\sin^2\alpha}{2g^2}$$

$$= \frac{u\sin\alpha^{\;2}}{2g}$$

7 As the maximum value of sine is 1 the maximum height will (unsurprisingly) be when α = 90°.

8 a The particle will hit the ground when y is equal to 0.

$$y = (u\sin\alpha)t - \frac{1}{2}gt^2 = t\left((u\sin\alpha) - \frac{1}{2}gt\right) = 0 \text{ when } t{=}0 \text{ or } t = \frac{2u\sin\alpha}{g}$$

b This is twice the time taken to reach maximum height so the particle takes the same length of time to descend as it does to ascend.

9 From the displacement equation the horizontal displacement (x) is $x = (u\cos\alpha)t$

Substitute in $t = \dfrac{2u\sin\alpha}{g}$ to get

$$x = \frac{2u^2\sin\alpha\cos\alpha}{g}$$

10 As the double angle formulae are not in the syllabus the best approach is to use a GDC to find the maximum angle, 45°.

11 This is a chance to use the formulae derived in question **9** to get

Range = 19.9 m

12 Using the formula from question **6** we obtain the maximum height = 2.87 m

13 Air resistance has been neglected.

14 **Factual:** What does a column vector describe in projectile motion?

Answer: Vectors can be used to describe projectile motion by specifying the horizontal and vertical components and integrating the acceleration equation.

15 **Conceptual:** How are vectors useful in describing projectile motion and how do we find the velocity equation?

Answer: Vectors can describe angular velocity in circular motion by differentiating the displacement equation, and acceleration can be found by differentiating the angular velocity equation.

Investigation 3

Conceptual understanding:

Vectors can describe angular velocity in circular motion by differentiating the displacement equation and acceleration can be found by differentiating the angular velocity equation.

Additional notes

This derives some of the general results for circular motion. The ideas are developed further in considering solutions of coupled differential equations in the next section.

1 Distance from origin is the magnitude of the displacement vector

$$= \sqrt{r^2\cos^2\omega t + r^2\sin^2\omega t} = r\sqrt{\cos^2\omega t + \sin^2\omega t} = r\,.$$

This is a constant value, hence the motion is on a circle with radius r.

2 **a** When t=0 the position vector of the particle is $\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} r \\ 0 \end{pmatrix}$ and hence its coordinates are (r, 0).

b The time it first returns to its starting position is equal to the period which is equal to $\dfrac{2\pi}{\omega}$. It could be mentioned that in physics and elsewhere ω (omega) is referred to as the angular velocity, though this term is not used in the course guide.

3 **a** The velocity is obtained by differentiating he displacement equation, hence

$$\boldsymbol{v} = \begin{pmatrix} -r\omega \sin \omega t \\ r\omega \cos \omega t \end{pmatrix}$$

b Speed is the magnitude of velocity and hence

$$\text{speed} = \sqrt{-r\omega \cos \omega t\ ^{2} + \ r\omega \sin \omega t\ ^{2}} = r\omega$$

c Show $\boldsymbol{r}.\boldsymbol{v} = 0$.

d The direction of motion is always tangential to the displacement vector. This can be illustrated by imagining an object on a string being spun in a circle and asking in which direction it would go if the string suddenly broke.

4 **a** The acceleration is found by differentiating the velocity vector $\boldsymbol{a} = \begin{pmatrix} -r\omega^2 \cos \omega t \\ -r\omega^2 \sin \omega t \end{pmatrix}$.

b Show $\boldsymbol{a}.\boldsymbol{v} = 0$.

c The modulus of $\boldsymbol{a}$ is given by

$$\boldsymbol{a} = \sqrt{-r\omega^2 \cos \omega t\ ^{2} + \ -r\omega^2 \sin \omega t\ ^{2}}$$

$$= \sqrt{r^2\omega^4 \ \cos^2 \omega t + \sin^2 \omega t} = r\omega^2$$

$$|\boldsymbol{a}| = r\omega^2 = \frac{r\omega\ ^{2}}{r} = \frac{|\boldsymbol{v}|^2}{r} \text{ from } \mathbf{3b}$$

d $\boldsymbol{a} = \ -r\omega \ \boldsymbol{r}$

5 **Conceptual:** How would you derive the velocity and acceleration equations for a particle moving in a circle from the general equation for its displacement?

Answer: Vectors can describe angular velocity in circular motion by differentiating the displacement equation and acceleration can be found by differentiating the angular velocity equation.

TOK

How do we relate a theory to the author? Who developed vector analysis, Josiah Willard Gibbs or Oliver Heaviside?

You might want to question this statement with – Who developed vector analysis, Josiah Willard Gibbs or Oliver Heaviside? Who else was involved?

How can you decide who to give credit to for a topic such as vector analysis?

Who decides?

These questions can be used to start a discussion that could end in a presentation or blog post.

Developing inquiry skills

Return to the opening problem for the chapter. The package will be dropped from the aircraft from a height of 150 m. The aircraft has a speed of 180 km h^{-1}. If air resistance can be ignored, find how far from the research station the package should be released:

a if the aircraft is flying horizontally

b if the aircraft is ascending at an angle of 30°.

Answer:

(let g = 9.8 m s^{-2})

a $\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$ Given that the horizontal velocity is initially 180 kmh^{-1} and the vertical velocity equal to zero and converting to ms^{-1} we obtain $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 50 \\ -9.8t \end{pmatrix}$.

We can choose the origin to at the point on the ground directly below the aircraft at the moment it releases the package – other positions for the origin are also possible. Hence initial displacement is (0, 150) and the package will hit the ground when y = 0.

The equation for displacement is found by integrating the velocity equation.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 50t \\ 150 - 4.9t^2 \end{pmatrix}$$

y = 0 when $150 - 4.9t^2 = 0 \Rightarrow t = 5.53$.

At this time x = 50 × 5.53 = 247 m. Hence the package should be released approximately 250 m from the research station.

b The initial velocity of the package is $\begin{pmatrix} 50\cos 30° \\ 50\sin 30° \end{pmatrix}$

hence $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 50\cos 30° \\ 50\sin 30° - 9.8t \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 43.3t \\ 150 + 25t - 4.9t^2 \end{pmatrix}$.

y = 0 when $150 + 25t - 4.9t^2 \Rightarrow t = 8.64$, and substituting in to the equation for x we obtain x = 374 m.

So the package should be released about 375 m from the research station. Which is about 50% further than if released when the aircraft was flying horizontally.

12.3 Exact solutions of coupled differential equations

Investigation 4

Conceptual understanding:

Eigenvalues and eigenvectors can be used to "decouple" a system to give equations in a single variable.

Additional notes

This is an important investigation as it guides the students through the steps of solving a coupled system of linear equations in the case of real distinct eigenvalues.

1 $x = 5e^{2t}$ which is found using the methods of Chapter 11.

2 This system of equations is solved by considering the two equations separately. This is possible as it is not a coupled system.

Eigenvectors allow us to **decouple** a system of coupled equations so they can be solved in a similar way.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3e^{2t} \\ 5e^{3t} \end{pmatrix}$$

This general formula for the solution of coupled linear differential equations is given in the formula book.

3 **a**

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

b $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ has eigenvalues of 2 and 3 and corresponding eigenvectors of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Hence answer to question **2** can be written as $\begin{pmatrix} x \\ y \end{pmatrix} = 3e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$..

Notice that the top line gives the first of the equations derived in question **2** and the bottom line gives the second one.

4 Differentiating $\dot{\boldsymbol{x}} = A\lambda_1 e^{\lambda_1 t}\boldsymbol{p}_1 + B\lambda_2 e^{\lambda_2 t}\boldsymbol{p}_2 = Ae^{\lambda_1 t}\ \lambda_1\boldsymbol{p}_1\ + Be^{\lambda_2 t}\ \lambda_2\boldsymbol{p}_2$

then, using the result that $\lambda_1\boldsymbol{p}_1 = \boldsymbol{M}\boldsymbol{p}_1$, this becomes

$$= Ae^{\lambda_1 t}\ \boldsymbol{M}\boldsymbol{p}_1\ + Be^{\lambda_2 t}\ \boldsymbol{M}\boldsymbol{p}_2\ = \boldsymbol{M}\ \ Ae^{\lambda_1 t}\boldsymbol{p}_1 + Be^{\lambda_2 t}\boldsymbol{p}_2\ = \boldsymbol{M}\boldsymbol{x}$$

5 Rate of growth of a substance is often proportional to the amount of substance present, hence $0.4x$ and $0.1y$ are both positive, 0.4>0.1 is consistent with the rate of growth of x being greater than of y. The growth of x is likely to decrease as y increases and comes into increasing contact with x, hence $-0.2y$. Y though benefits from an increased amount of X hence $0.1y$.

6

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0.4 & -0.2 \\ 0.1 & 0.1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Multiplying out the right-hand side gives the original equations.

7 First we need to find the eigenvalues and eigenvectors

$$\begin{vmatrix} 0.4-\lambda & -0.2 \\ 0.1 & 0.1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 0.5\lambda + 0.06 = 0 \Rightarrow \lambda = 0.2,\ 0.3$$

Corresponding eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Hence using the general solution,

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{0.2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{0.3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Putting in initial conditions $\begin{pmatrix} 10 \\ 6 \end{pmatrix} = \begin{pmatrix} A \\ A \end{pmatrix} + \begin{pmatrix} 2B \\ B \end{pmatrix}$.

$A = 2,\ B = 4$

$$\begin{pmatrix} x \\ y \end{pmatrix} = 2e^{0.2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 4e^{0.3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

8 **a** In the long term the second term will dominate and so X will cover approximately twice as much of the tree as Y. If values are found by putting the equations into the GDC and using the Table function it can be seen that the ratio is 2.0 to 2 significant figures after 23 weeks.

b At some point there will be no space to expand on the tree as the coverage is exponential. The weather will likely change and the growths change with it. It takes about 22 weeks for the coverage to be about 1m^2 , so the model might be valid to this point.

9 **Conceptual:** How are eigenvalues and eigenvectors useful when solving a system equations?

Answer: The two equations obtained for x and y are given in terms of a single variable.

Hence the conceptual understanding for this investigation:

Eigenvalues and eigenvectors can be used to "decouple" a system to give equations in a single variable.

Investigation 5

Conceptual understanding:

A phase portrait displays geometrical representations of future paths, in other words how different variables change over time with different initial values and they can depict the solutions of differential equations.

Additional notes

This investigation introduces the idea of a phase portrait and some of its key features.

General solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + Be^{5t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

1 **a**

The eigenvectors are $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

The Cartesian equations can be seen easily from a sketch but algebraically they can convert to a parametric equation and hence to the Cartesian form.

$\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, , hence $x = \lambda$ and $y = 2\lambda$. If λ is eliminated you get $y = 2x$.

b

Let $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 14 \end{pmatrix}$. when $t = 0$ substituting into

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + Be^{5t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix} = A\begin{pmatrix} 1 \\ 2 \end{pmatrix} + B\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$2 = A - B$$
$$4 = 2A + B$$

Hence

$$A = 2,\ B = 0\quad \begin{pmatrix} x \\ y \end{pmatrix} = 2e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Whatever initial value is chosen on $y = 2x$ the solution will be of the form

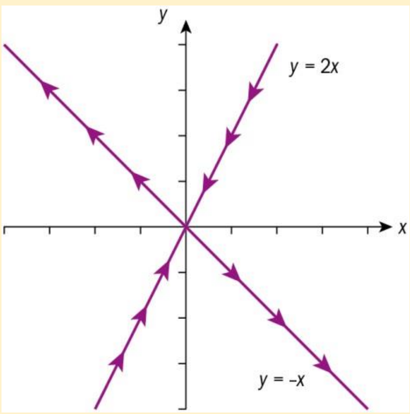
$$\begin{pmatrix} x \\ y \end{pmatrix} = k_1 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

c In a similar way we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = k_2 e^{5t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

d Whatever initial value is chosen the solution will only have one term, and so the motion will remain on this line. In the first case the values will go towards the origin as t increases, and in the second the movement will be away from the origin.

2 **a** and **b**



3 **a**

For large values of t,

$$\begin{pmatrix} x \\ y \end{pmatrix} \approx k e^{5t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

as the term containing the component with the largest exponent will dominate as t increases.

b For large negative values of t,

$$\begin{pmatrix} x \\ y \end{pmatrix} \approx k e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

c Solving

$$\begin{pmatrix} 6 \\ 6 \end{pmatrix} = A \begin{pmatrix} 1 \\ 2 \end{pmatrix} + B \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

gives

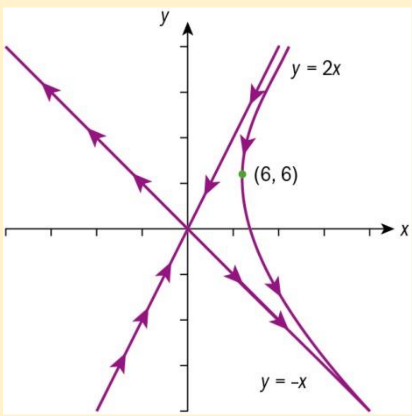
$$\begin{pmatrix} x \\ y \end{pmatrix} = 4e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2e^{5t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

d From the table students should notice that as t increases in a positive direction the values tend towards a ratio of $1:-1$ and as t increases in the negative direction $1:2$. Also significant for the sketch is that the x values stop decreasing around $t = -0.2$.

X	Y1	Y2		
-0.3	5.8457	10.353		
-0.1	5.6214	9.0355		
-0.1	5.6337	7.6283		
0	6	6		
0.1	6.9168	3.9413		
0.2	8.7115	1.1133		
0.3	11.927	-3.037		
0.4	17.459	-9.416		
0.5	26.791	-19.51		
0.6	42.366	-35.78		
0.7	68.217	-62.26		

X = 0.7

e



f This can be done in a similar way to the previous trajectory, but if plotting several trajectories then it will be simpler to use Geogebra (or similar software). In Geogebra the equation can be entered as a parametric equation as follows

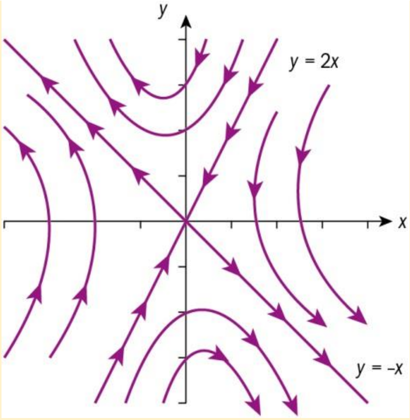
Curve(a*e^(-t)-b*e^(5t), 2a*e^(-t)+b*e^(5t), t, -10, c)

a, b and *c* are sliders. *c* gives the maximum value of *t* so moving the slider from left to right will show the direction of the trajectory.

If initial values of *x* and *y* are preferred to *a* and *b* then students could derive the fact that if the initial value is $\left(x_0, y_0\right)$ then *a* and *b* are the solutions to $\begin{aligned} x_0 &= A - B \\ y_0 &= 2A + B \end{aligned}$ which are

$\dfrac{x_0 + y_0}{3}$ and $\dfrac{y_0 - 2x_0}{3}$.

There are also several online sites which will allow initial values to be entered directly and the phase portraits seen.



4 **a** $y = 2x,\ y = -\frac{1}{2}x$,

b It stays on the line but moves away from the origin. This is because:

- the second coefficient will be zero, so it will stay on the line,
- the exponent is positive so it will move away from the origin.

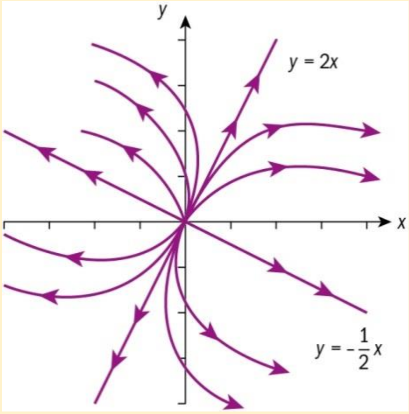
The fact that initial values that lie on lines through the origin in the direction of the eigenvectors will stay on these lines should be learned to help with drawing phase portraits.

c The dominant term for large values of t is $Be^{3t}\begin{pmatrix}2\\-1\end{pmatrix}$ so the gradient of the trajectory approaches $-\frac{1}{2}$ hence its direction is parallel to $\begin{pmatrix}2\\-1\end{pmatrix}$.

d (0, 0)

e For large negative values of t the dominant term is $Ae^{t}\begin{pmatrix}1\\2\end{pmatrix}$ so the gradient will be approximately 2.

f



g The diagram would be the same except the arrows would be reversed.

5 **Conceptual:** Why are phase portraits a useful way of depicting the solutions to differential equations?

Answer: A phase portrait displays geometrical representations of future paths, in other words how different variables change over time with different initial values and they can depict the solutions of differential equations.

TOK

"There is no branch of mathematics, however abstract, which may not someday be applied to phenomena of the real world." – Nikolai Lobatchevsky

Where does the power of mathematics come from? Is it from its ability to communicate as a language, from the axiomatic proofs or from its abstract nature?

A great chance for academic students to experience the complexities of mathematics. The nature of mathematics. reveals hidden patterns that help us understand the world around us.

Research an abstraction in mathematics, axioms and mathematical language.

Vectors treats abstract objects, mathematics often relies on reason but there is a place for sense perception, imagination, and even intuition.

Investigation 6

Conceptual understanding:

A matrix for a coupled system with imaginary eigenvalues indicates periodic motion, while complex eigenvalues display motion forming a spiral.

Additional notes

This investigation is to lead students to work out the phase portrait when the eigenvalues are complex. The syllabus does not require detailed knowledge of the theory behind the paths and the students are certainly not expected to know how to find complex eigenvectors or solutions to the system. They only need to be able to say whether the trajectories spiral towards or away from the origin (or around it in the case of imaginary eigenvalues), and whether the motion is clockwise of counter-clockwise.

1 **a** The eigenvalues are 2i and -2i.

b $b=2$, $e^{2ti} = \cos 2t + i\sin 2t$

c $\boldsymbol{x} = Ae^{2ti}\boldsymbol{v}_1 + Be^{-2ti}\boldsymbol{v}_2$

It should be noted that both *A* and *B* and the eigenvectors could contain imaginary values.

Writing in trigonometric form we have

$\boldsymbol{x} = A\ \cos 2t + i\sin 2t\ \boldsymbol{v}_1 + B\ \cos\ -2t\ + i\sin\ -2t\ \ \boldsymbol{v}_2$

It can be seen that that the period of each of the trigonometric functions is $\frac{2\pi}{2} = \pi$ and hence the coordinates will again be equal to their initial value when $t = \pi$.

2 **a** Solving the characteristic equation gives complex roots and hence

A eigenvalues are $-1 \pm 2\mathrm{i}$.

B eigenvalues are $2 \pm \mathrm{i}$.

b The real parts are equal for both eigenvalues. The imaginary parts have the same magnitude but different sign.

c Complex solutions to quadratic equations will always be conjugate pairs.

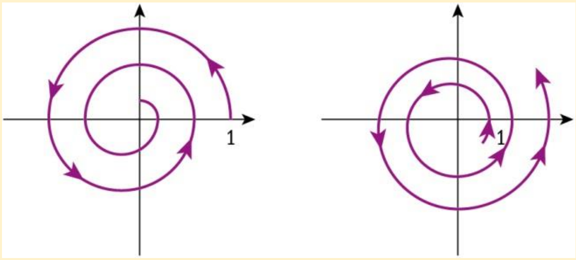
3 **a** If $a < 0$ the exponential part will tend to zero with increasing *t*

b For A $\frac{dy}{dt} = 2x + y$ and at (1, 0) $\frac{dy}{dt} = 2$ so as *t* increases *y* is increasing so the motion is counter clockwise.

For B at (1, 0) $\frac{dy}{dt} = 1$ so as *t* increases *y* is increasing so the motion is counter-clockwise.

c Using $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}$ gives the gradient for system A at (1,0) as $-\frac{2}{3}$ and for B as $\frac{1}{2}$.

d The easiest way to draw the phase portrait is to begin with the point chosen, (1, 0) in this question, and then spiral in or out depending, on the value of *a,* and in the direction deduced from consideration of the gradients.



4 **Conceptual:** How is the motion of a particle different when the matrix for a coupled system has either imaginary or complex eigenvalues?

Answer: A matrix for a coupled system with imaginary eigenvalues indicates periodic motion, while complex eigenvalues display motion forming a spiral.

12.4 Approximate solutions to coupled linear equations

Investigation 7

Conceptual understanding:

A predator–prey model normally displays a periodic rise or fall of each population and can be developed from a system of coupled differential equations which can be solved using numerical methods.

Additional notes

This investigation introduces the student to the predator–prey model and some methods for interpreting the real-life implications of coupled differential equations.

The following two GeoGebra sites will help illustrate some of the ideas behind the model. There a several other sites also available that simulate this problem. Note that the site given below uses approximation methods so the results after the first cycle are not periodic. It also demonstrates how reducing the step size improves accuracy at the expense of time to do the calculations.

https://www.geogebra.org/m/FwQfAxqE

In addition, the TED talk *Math can help uncover cancer's secrets* is an excellent introduction to the applications of these models. Indeed it would make a good introduction at the start of the course to the general ideas of mathematical modeling.

https://www.ted.com/talks/irina_kareva_math_can_help_uncover_cancer_s_secrets#t-446541

1 As the population of the prey increases so does the population of the predator, hence the *xy* is positive. Ultimately, though, as *y* increases eventually there will be too many for the prey available so the population of prey is likely to begin to decrease and when it gets below a certain point the population of *y* will also begin to decrease.

2 The two equilibrium points are where

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

$3x\ \ 1-y\ \ =0,\quad y\ \ x-2\ \ =0$

Both are equal to 0 when x = 0, y = 0.

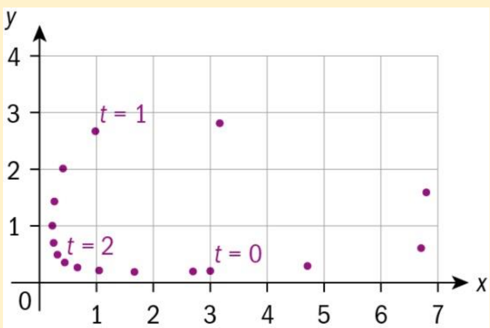
When y = 1, x = 2.

3 The results are given here to ensure that the student is working with the correct equations, such precision would not normally be appropriate for a solution obtained using the Euler method.

The student needs to know how to do these calculations on a GDC for the exam.

It would though be a useful exercise to also do this work on a spreadsheet. Plotting the graph in question **6** is a useful exercise in demonstrating the standard predator–prey graph.

4 **a** Note that taking a larger step length leads to unstable results.

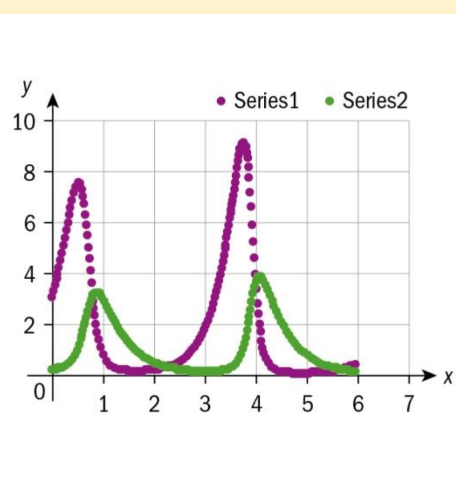


b It is likely that the equilibrium point is a center as the trajectories go round it.

5 Population of fish start to fall when $\dfrac{dx}{dt}=0$ which means y =1. If y =1 then $\dfrac{dy}{dt}=x-2$, as x is much bigger than 2 the shark population will continue to grow until the fish drop down to a population of about 2000.

6 The peaks and troughs of both curves follow the same pattern with the growth of the sharks following that of the fish after a period of about 4 months.

t	x	y
0	3	0.2
0.02	3.144	0.204
0.04	3.294157	0.208668
0.06	3.450564	0.214068
0.08	3.613278	0.220279
0.1	3.782319	0.227386
0.12	3.957656	0.235492
0.14	4.139195	0.244712
0.16	4.326772	0.255182
0.18	4.520132	0.267057
0.2	4.718912	0.280517
0.22	4.922622	0.295771
0.24	5.130621	0.31306
0.26	5.342087	0.332661
0.28	5.555986	0.354897
0.3	5.771037	0.380137
0.32	5.985672	0.408807
0.34	6.197994	0.441395



7 **Conceptual:** How can we develop a predator–prey model and what can the model display?

Answer: A predator–prey model normally displays a periodic rise or fall of each population and can be developed from a system of coupled differential equations which can be solved using numerical methods.

Developing inquiry skills

A plane needs to deliver a supply package to a polar research station. The package will be dropped from the plane from a height of 150 m. The plane has a speed of 180 km h^{-1} and is flying horizontally. Air resistance acts on the package in such a way that the horizontal acceleration ($\ddot{x}$) and the vertical acceleration ($\ddot{y}$) are given by the following equations:

$$\ddot{x} = -0.50\dot{x} \quad \ddot{y} = -9.8 + 0.02\dot{y}^2$$

a **i** Use the substitution $u = \dot{x}$ to write $\ddot{x} = -0.05\dot{x}$ in terms of u.

ii Hence write down an equation for u in terms of t, the time in seconds from when the package was dropped.

iii Find an equation for x, the horizontal distance travelled by the package.

iv If the package could be released from any height what would be the maximum horizontal distance from the research station that the package could be released.

b Use Euler's method with a step length of 0.1 seconds to find:

i the time at which the package will hit the ground to the nearest tenth of a second

ii the vertical speed at which the package will hit the ground.

c Hence find:

i the distance from the research station at which the package should be released

ii the speed (the magnitude of the resultant of horizontal and vertical velocity) at which the package hits the ground.

Answer:

a **i** Though the substitution is not necessary it does clarify the differential equation that needs to be solved.

$$\dot{u} = -0.05u$$

ii The command term is 'write down' because students at this stage of the course should recognize the equation for exponential growth. The student may remember from the previous occasion this question was considered that 180 kmh^{-1} is equivalent to 50 m s^{-1}. If not this would need to be calculated.

$$u = 50e^{-0.05t}$$

iii

$$\dot{x} = 50e^{-0.05t}$$

$$x = c - \frac{50}{0.05}e^{-0.05t} = c - 1000e^{-0.05t}$$

At $t = 0$, $x = 0$ hence

$$x = 1000\ 1 - e^{-0.05t}$$

iv Distance covered never gets greater than 1000 m, so this is the maximum distance from the research station it could be dropped.

b This part has less scaffolding as the work done in the chapter should be sufficient preparation. The differential equation could be solved directly using partial fractions but this technique is outside the syllabus.

Let $\dot{y} = v$ so $\dot{v} = -9.9 + 0.02v^2$.

The equations are

$$y_{n+1} = y_n + 0.1v_n$$
$$v_{n+1} = v_n + 0.1\ \ -9.8 + 0.02v_n^2$$

i 8.4 seconds

ii 22.1 m s^{-1}

c **i** Substitute 8.4 seconds into the equation $x = 1000(1 - e^{-0.05t})$ to get x = 343 m

ii Using $\dot{x} = 50e^{-0.05t}$ horizontal speed is 32.85 m s^{-1}.

The two velocities can be written as $\begin{pmatrix} 0 \\ -22.1 \end{pmatrix}$, $\begin{pmatrix} 32.85 \\ 0 \end{pmatrix}$

Hence the resultant is $\begin{pmatrix} 32.85 \\ -22.1 \end{pmatrix}$ and the speed is equal to $\sqrt{32.85^2 + 22.1^2}$ = 39.6 m s^{-1}

This result can come directly from a diagram and use of Pythagoras theorem.

Disease modelling

Approaches to Learning/learner profile: Communication, Critical Thinking

Exploration Criteria: Mathematical Communication (B); Reflection (D); Use of Mathematics (E)

IB Topic: Differential Equation (variables separable)

Introduction

This task asks the students to develop a model for a simple case of an infectious disease. They will be looking at producing a general solution to a differential equation in which the variables are separable and then simulating a population to see how closely these results follow this general solution. There are opportunities to reflect on the assumptions made in the model and then to reflect on any differences between the simulated results and the expected results.

Imagine a population threatened by an infectious disease.

Within the population there are two groups, those that are infected (I) and those that are healthy (H).

Assume that each year, the probability that a healthy person catches the disease is c, and the probability that an infected person recovers is r.

Let x be the proportion of the population that are infected and t be the number of the years from the beginning of recording.

Show that a differential equation that will model the rate of change of the proportion of the population affected with respect to time is given by:

$$\frac{dx}{dt} = c(1 - x) - rx$$

$c(1 - x)$ is the probability of a healthy person being infected multiplied by the proportion of the population who are healthy and rx is the probability of an infected person recovering from the disease multiplied by the proportion of the population who are infected. The difference between these will therefore be the rate of change of the proportion of infected people with respect to time, that is $\frac{dx}{dt}$.

Reflect on the assumptions made in this hypothetical situation and how they may differ in a real-life situation involving an infectious disease in a population.

This is a very simplified version of an infectious disease. It assumes that a person does not get infected or recovers more than once within a year. It assumes that these rates remain constant over time and that no cure is found. It also assumes that no-one dies or leaves or arrives in the population during the time being considered.

By separating the variables find the general solution of this differential equation:

$$\frac{dx}{dt} = c(1-x) - rx = c - (r+c)x$$

Separating the variables

$$\frac{1}{c-(r+c)x}dx = dt$$

$$\Rightarrow \frac{-1}{r+c}\ln(c-(r+c)x) = t + a$$

$$\Rightarrow \ln(c-(r+c)x) = -(r+c)(t+a)$$

$$\Rightarrow c-(r+c)x = e^{-(r+c)t-(r+c)a}$$

$$\Rightarrow (r+c)x = c - e^{-(r+c)t-(r+c)a}$$

$$\Rightarrow x = \frac{c}{r+c} - \frac{1}{r+c}e^{-(r+c)t-(r+c)a}$$

$$\Rightarrow x = \frac{c}{r+c} - be^{-(r+c)t}$$

Using this general solution what would you expect to happen to the proportion of people affected by the disease over time? (i.e. what happens as t tends towards infinity?)

As t tends towards infinity x will tend towards $\frac{c}{r+c}$.

Simulation

You are now going to **simulate a specific case of this situation**.

Assume that $c = 0.3$ and $r = 0.2$.

Use these values of c and r to simulate the proportion of infected people over time

To do this assume a population size of 10 of which, in year 0, 7 are healthy ($H = 7$) and 3 are infected with a particular disease ($I = 3$).

In year 0, therefore, $x = 0.3$

We will simulate what happens to each of the 10 people over the course of 10 years.

Consider the first person (who starts off healthy in year 0). Generate a series of 10 random numbers from a list of numbers from 1 to 10.

If the number is a 1, 2 or 3 and the person is healthy ($c = 0.3$) then they will catch the disease; otherwise they will remain healthy.

This corresponds to $c = 0.3$

If the number is a 1 or a 2 and the person is infected they will recover; otherwise they will remain infected

This corresponds to $r = 0.2$

Example

Year	0	1	2	3	4	5	6	7	8	9	10
Random number		5	3	7	3	8	2	5	5	9	6
Healthy or infected?	*H*	*H*	*I*	*I*	*H*	*H*	*I*	*I*	*I*	*I*	*I*

Conduct this simulation 10 times for each of the 10 people in your population

Students may want to use the website www.random.org to generate a list of random integers from 1 to 10 or they could use their calculator or write a code or create a spreadsheet that could do this for them.

For each year calculate the value of x, the proportion of people infected. Plot a graph of x against time?

Here is a partially filled table to give an idea:

1	*H*	*H*	*I*	*I*	*H*	*H*	*I*	*I*	*I*	*I*	*I*
2	*H*	*I*	*H*	*H*	*H*	*I*	*H*	*H*	*I*	*I*	*H*
	…	…	…	…	…	…	…	…	…	…	
10	*I*	*I*	*I*	*H*	*H*	*H*	*H*	*H*	*I*	*H*	*I*
x	0.3	0.4	0.5	0.5	0.5	0.6	0.6	0.5	0.6	0.6	0.6

What does the graph suggest will happen as x tends to infinity?

The graph should approach a horizontal asymptote (of $x = 0.6$ (that is $\frac{c}{r+c}$ where $c = 0.3$ and $r = 0.2$). If it is not close, then reflect on the small sample size. The data would be more reliable if the population were larger. This could be achieved by pooling all the data in the class or using a code or a spreadsheet to generate more data.

Now consider the differential equation again.

Rewrite the equation for $\frac{\mathrm{d}x}{\mathrm{d}t}$ but use the specific values of $c = 0.3$ and $r = 0.2$ from before

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.3(1-x) - 0.2x = 0.3 - 0.5x$$

Separating the variables:

$$\frac{1}{0.3+0.5x}\mathrm{d}x = \mathrm{d}t$$

$\Longrightarrow \dfrac{-1}{0.5}\ln(0.3 - 0.5x) = t + a$

$\Rightarrow -2\ln(0.3 - 0.5x) = t + a$

$\Longrightarrow \ln(c - (r + c)x) = -(r + c)(t + a)$

$\Longrightarrow 0.3 - 0.5x = e^{-0.5t-0.5a}$

$\Longrightarrow 0.5x = 0.3 - e^{-0.5t-0.5a}$

$\Longrightarrow x = \dfrac{0.3}{0.5} - 2e^{-0.5t-0.5a}$

$\Longrightarrow x = 0.6 - be^{-0.5t}$

Solve this differential equation by separating the variables and using the starting conditions of t = 0 and x = 0.3 to find the value of b in the equation.

if $x = 0.6 - be^{-0.5t}$

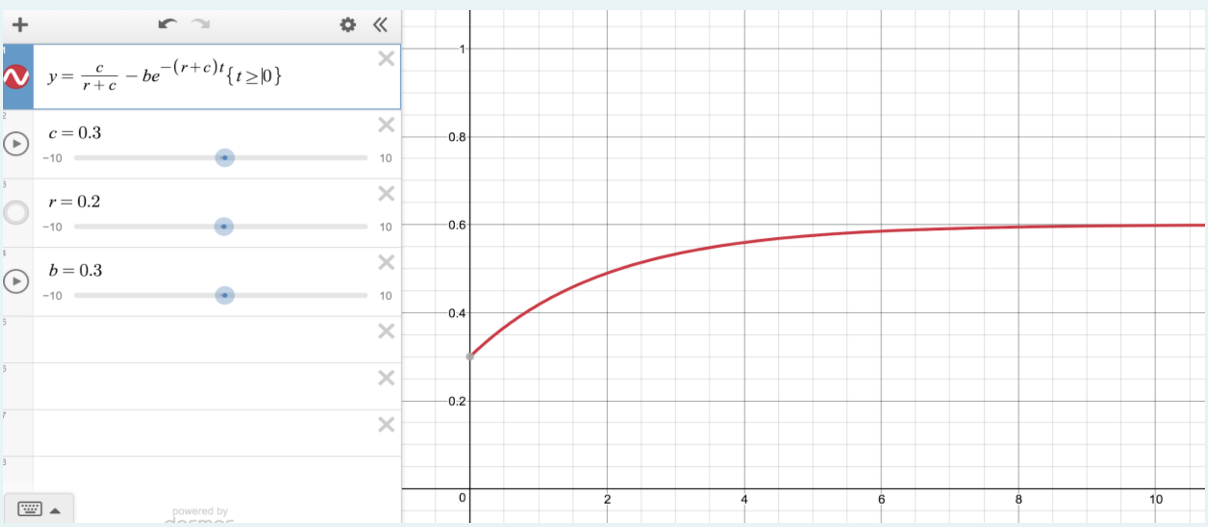
and

$x = 0.3$ when $t = 0$

$\Longrightarrow 0.3 = 0.6 - b^0$

so $b = 0.6 - 0.3 = 0.3$

Sketch the graph of x against t.



What happens as t tends to infinity?

As t tends to infinity x approaches 0.6.

(note in Desmos y has been used in the place of x. Note also the restricted domain of $t \geq 0$)

How well does this fit with your data in your simulation?

This is another opportunity for reflection on sample size, etc.)

Extension

What happens to the solution if you vary c and r?

What happens to the solution if you vary the starting conditions of the proportion of infected people in the population?

It would be useful to draw the graph of the general solution and use “sliders” to vary these values and observe what happens This can be done easily on Desmos or autograph or other graphing packages.

In the second question it would be necessary to ensure that the value of b also varies as the values of r and c vary and the starting values vary.

13 Representing multiple outcomes: random variables and probability distributions

Essential understandings

Statistics is concerned with the collection, analysis and interpretation of quantitative data and uses the theory of probability to estimate parameters, discover empirical laws, test hypotheses and predict the occurrence of events. Statistical representations and measures allow us to represent data in many different forms to aid interpretation.

Probability enables us to quantify the likelihood of events occurring and so evaluate risk. Both statistics and probability provide important representations which enable us to make predictions, valid comparisons and informed decisions. These fields have power and limitations and should be applied with care and critically questioned, in detail, to differentiate between the theoretical and the empirical/observed. Probability theory allows us to make informed choices, to evaluate risk and to make predictions about seemingly random events.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Modelling and finding structure in seemingly random events facilitates prediction.
- Different probability distributions provide a representation of the relationship between the theory and reality, allowing us to make predictions about what might happen.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	Investigation
A discrete probability function can be found as the generalisation of a random process.	Investigation 1
The expected value of a discrete random variable predicts the likely average of the values of the random variable in a number of trials.	Investigation 2
The binomial distribution describes situations for discrete random variables with two possible outcomes, a finite number of trials and a constant probability of success/failure.	Investigation 4
The Poisson process/distribution models situations with an average number of successes in a given interval of time or space.	Investigation 6
The parameter in the Poisson distribution models or can alter the mean and variance.	Investigation 7
Naturally occurring symmetric data where very small and very large data points are relatively rare can be modelled by the normal distribution.	Investigation 8
The parameters of the normal distribution function model the mean and variance in normally distributed data.	Investigation 9
The normal cumulative distribution function predicts probabilities for a given interval of the random variable, and a sketch of the area can support the evaluation of the probability.	Investigation 10

Conceptual understandings (cont.)	**Investigation**
A sum of two independently distributed Poisson random variables follows a Poisson distribution.	Investigation 11
Linear combinations of *n* independent normal random variables lead to a normal distribution and the mean and variance can be found from the formulae for the mean and variance of linear combinations of *n* independent random variables.	Investigation 12
The distribution of sample means of *n* independent normally distributed random variables allows predictions to be made about the central tendency of a sample.	Investigation 13
The central limit theorem allows predictions to be made about the central tendency of a sample even with an unknown distribution of the underlying population.	Investigation 14

Syllabus sections covered in this chapter:

- SL4.2*
- SL4.7*
- SL4.8*
- SL4.9*
- AHL4.17
- AHL4.14
- AHL4.15

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 571	Page 589: Example 7 Page 607: Example 16	Page 582: Example 4 Page 584: Example 5 Page 589: Example 6 Page 589: Example 7 Page 591: Example 8 Page 596: Example 9 Page 598: Example 11 Page 599: Example 12 Page 612: Example 18	Pages 579, 586, 591, 600, 603, 613

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 614	Page 616	N/A

Developing inquiry skills

I woke with a premonition: tomorrow my café would be inspected by the hygiene authority. I knew 3 months ago to expect a visit within a year, but not *when*. I'd expected longer than 3 months to prepare. I set out to arrive at the cafe early, but had to wait for 19 minutes at the bus stop. I usually have to wait 5 minutes for a bus on average. Then 3 came along at once!

On arrival I checked the machines. Two of the four percolators on the espresso machine were not functioning! Was this bad luck, I wondered? I knew that each percolator had a one in a thousand chance of not functioning on any given day ⋯ I started to think about this problem but decided to focus on fixing the percolators.

I had to open the espresso machine combination lock to access the percolators. I was given the 3-digit code years ago but had lost it! I could remember that the first two digits were the same. I decided to guess. After 34 trials I found the right code and the door swung open. I fixed the percolators: but did I have enough coffee? I knew the average weight of a bag was 3 kg and that it could range from 2.8 to 3.2 kg … how sure could I be that I would not run out? Matt the barista rushed in. "Four inspectors came!" he exclaimed. "Why the past tense?" I enquired. "They were too heavy for the maximum elevator load of 300 kg and they didn't have time to use the stairs" came the answer … time to relax with an espresso I figured.

- How many situations are there in this tale that involve probability?
- What kind of variables are involved?

- How could you represent and quantify the variables?
- Write down examples where chance has played a role in *your* life

Answer:

- There are six situations: the day the inspectors arrive, the number of buses arriving at the bus stop, the length of time spent waiting at the bus stop, the number of failing percolators, the number of trials carried out in order to find the correct combination and the total weights of the inspectors are all situations that involve probability.
- These situations involve variables are either found by counting or by measuring.

You can represent the

- day of arrival of the inspectors as a positive integer with the sample space of all the days Chancer's café is open for the year.
- length of time spent waiting for a bus as a real number that is at least zero.

13.1 Modelling random behaviour

Investigation 1

Conceptual understanding:

A discrete probability function can be found as the generalization of a random process.

Additional notes

Starting this investigation by throwing five dice with the students is an engaging way to begin, as is watching some of the many "Yahtzee" clips found online. Pressing F9 on the spreadsheet enables the student to experience the variability of random behaviour, as they will be able to see anomalies, and also behaviour that corresponds to the expected value given in the answer to question **3**.

1 The sample space is $\mathbb{Z}^+$.

2 $\frac{1}{1296}$

3 1296

4
$$\frac{X_1 + X_2 + \ldots + X_n}{n} \sim N\left(\frac{n\mu}{n}, \frac{n\sigma^2}{n^2}\right)$$
$$\Rightarrow \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

5 They should add to 1 because they represent the probabilities of each and every member of the sample space.

6 1 can be found with application of geometric series.

7 **Conceptual:** How can a discrete probability distribution function be found?

Answer: A discrete probability function can be found as the generalization of a random process.

TOK

"Those who have knowledge, don't predict. Those who predict, don't have knowledge." – Lao Tzu.

Why do you think that people want to believe that an outside influence such as an octopus or a groundhog can predict the future?

Background research on "Paul the octopus" and "Groundhog Day" where Punxsutawney Phil is a groundhog, immortalised in the film Groundhog Day", can help produce interesting discussions.

Do you believe that people can predict the future with certainty? Can you give examples? If so, why do you believe this?

Which of the following are involved in prediction and how? Reason, intuition, emotion, faith.

Investigation 2

Conceptual understanding:

The expected value of a discrete random variable predicts the likely average of the values of the random variable in a number of trials.

Additional notes

Using the spreadsheet to generate many trials is an excellent way for students to experience variability and anomalies in random behaviour, and in doing so they will automatically compare experimental data with the predictions made in parts a and b.

1 Approach **a** works quickly and is reliable in a symmetric distribution but may not be precise. Approach **b** is precise and rigorous but takes time. **c** is convincing because you can see confirmation from real data, but the data can vary.

2 **Conceptual:** What does the expected value of a discrete random variable predict about the outcomes of a number of trials?

Answer: The expected value of a discrete random variable predicts the likely average of the values of the random variable in a number of trials.

Developing inquiry skills

Return to the opening problem.

Which of the situations in the café involve discrete probability distributions?

Answer: The day on which the inspector arrives, the number of buses arriving at the bus stop, the number of failing percolators, the number of trails needed before finding the correct combination.

13.2 Modelling the number of successes in a fixed number of trials

TOK

Is it possible to reduce all human behaviour to a set of statistical data?

A class discussion might go like this:

Social scientists use statistics to study human behaviour, for example, population, income, birth rates. Can you think of some more examples of this?

The United Nations and World Health Organization collect data and use it to help plan aid and development programs.

Can you think of some aspects of human behavior from other areas of knowledge that cannot be measured?

Investigation 3

This makes a link to the work of Kahneman and Tversky, and should open up discussion about commonly help misconceptions and biases.

Students may arrive at the understanding that although HTHTTH is as likely an outcome as THTTTT, it is more likely to obtain a *total* of three heads in six trials than a total of one head, because there are more elements in the sample space of the event "throw a total of three heads" than in "throw one head in six trails". This leads into the question of how to calculate (count) the number of elements in the sample space. Students can reflect on the need to quantify the sample space before making a subjective or a theoretical approach to finding a probability.

Investigation 4

Conceptual understanding:

The binomial distribution describes situations for discrete random variables with two possible outcomes, a finite number of trials and a constant probability of success/failure.

Additional notes

Students who wish to explore the study of probabilistic judgments in psychology (the representativeness fallacy for example) have an introduction to this idea in question **7** and **8** of this investigation, Investigation 3 and also the opening problem of Chapter 5.

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

x	0	1	2	3	4
$P(X = x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

1 The numbers in each row of Pascal's triangle appear in the probability distribution tables as the numerators of the unsimplified fractions.

2 These numbers represent the number of ways an outcome can happen.

3

x	0	1	2	3	4	5
$P(X = x)$	$\left(\frac{1}{2}\right)^5$	$5\left(\frac{1}{2}\right)^5$	$10\left(\frac{1}{2}\right)^5$	$10\left(\frac{1}{2}\right)^5$	$5\left(\frac{1}{2}\right)^5$	$\left(\frac{1}{2}\right)^5$

4 Yes

5 **Factual:** Complete: "Let X be the discrete random variable equal to the number of heads tossed in n trials of a fair coin.

Then $\mathrm{P}(X = x) = {}^{n}C_{x}\left(\frac{1}{2}\right)^{x}$ for $x \in \{0,1,\ldots n\}$"

6 **Factual:** The experiment is changed so that the coin is not fair and it is thrown 5 times.

P(H) = p, P(T) = 1 − p,.

Answer: $\mathrm{P}(X = x) = {}^{5}C_{x}p^{x}(1-p)^{5-x}$ for $x \in \{0,1,2,3,4,5\}$.

7 0.3125

8 0.015625

9 **Factual:** Which part of the formula for the binomial distribution counts possibilities?

Answer: The binomial coefficient.

10 **Conceptual:** What situations are described by binomial distributions?

Answer: The binomial distribution describes situations for discrete random variables with two possible outcomes, a finite number of trials and a constant probability of success/failure.

Investigation 5

1 The closer the probability of success is to 0.5, the greater the spread.

2 Increasing the number of trials will make the spread greater.

3 Both the parameters n and p affect Var(X).

TOK

What does it mean to say that mathematics can be regarded as a formal game lacking in essential meaning?

This can lead to a debate about what is a game and what is mathematics.

What is mathematics?

"I think many physicists, including myself, agree that there should be some complete description of the universe and the laws of nature. Implicit in that assumption is the universe is intrinsically mathematical." – Simeon Hellerman

What is mathematics? "Very simple: mathematics is the science of structure, order, and relation that has evolved from elemental practices of counting, measuring, and describing the shapes of objects. It deals with logical reasoning and quantitative calculation, and its development has involved an increasing degree of idealization and abstraction of its subject matter. Since the 17th century, mathematics has been an indispensable adjunct to the physical sciences and technology, and in more recent times it has assumed a similar role in the quantitative aspects of the life sciences". – Jorge Morales Pedraza

13.3 Modelling the number of successes in a fixed interval

Investigation 6

Conceptual understanding:

The Poisson process/distribution models situations with an average number of successes in a given interval of time or space.

Additional notes

Students can be encouraged to offer their own examples in class discussion.

1 **Factual:** What are all the similarities and differences between the random variables *L* and *U*?

Answer: *L* and *U* are both discrete random variables which assign probabilities to events relating to Nicola's punctuality. *L* follows a binomial distribution because it has a fixed number of trials and a constant probability of success. However, *U* does not have a fixed number of trials nor does it have a fixed probability of success – instead it depends on the rate at which buses arrive at the stop.

2 **Factual:** For the following random variables, which ones can be modelled by the Poisson and which by the binomial?

Answer: Poisson: A, C, D, F, Binomial: B, E

3 **Conceptual:** What situations does the Poisson process model?

Answer: The Poisson process/distribution models situations with an average number of successes in a given interval of time or space.

4 **Conceptual:** How do the parameters and sample spaces of the Poisson distribution and the binomial distribution compare and contrast?

Answer: The binomial has two parameters – a fixed number of trials and a probability of success. Therefore, the sample space of the binomial is a finite set of consecutive members of $\mathbb{N}$. The Poisson has only one parameter which is a rate, not a probability. Also, the sample space of the Poisson is an infinite set of consecutive members of $\mathbb{N}$.

Investigation 7

Conceptual understanding:

The parameter in the Poisson distribution models or can alter the mean and variance.

Additional notes

Having students alter the value of the parameter on the spreadsheet is key, so they can experience when the distribution is skewed and when it is symmetric.

1 Yes, the result is accurate provided that only negligible probabilities of your distribution lie outside the 30 values. If the parameter was 30, for example, the result would not be accurate.

2 The shape is skewed when the size of the parameter is small. As the parameter increases in size, the distribution becomes more symmetrical.

3 **Factual:** How does the mean and the variance of the distribution change when the parameter is changed?

Answer: Both the mean and the variance of the distribution appear to be proportional to the parameter.

4 **Conceptual:** How can you infer directly from the definition of the Poisson model that the parameter is the mean?

Answer: The Poisson parameter is the average number of successes in a given interval of time or space.

5 The greater the mean, the more spread the data.

6 **Conceptual:** What does the parameter of the Poisson distribution model?

Answer: The parameter in the Poisson distribution models or can alter the mean and variance.

TOK

A model might not be a perfect fit for a real-life situation, and the results of any calculations will not necessarily give a completely accurate depiction. Does this make it any less useful?

We often use a theoretical distribution, such as the binomial theorem, to describe a random variable that occurs in a real-life situation. This process is called modelling and enables us to make calculations and possibly predict.

However, it is unusual for a model to fit a real-life situation perfectly and we have to be ready for some error, which is a normal situation in life. However, we can often give a percentage chance of an event like forecasting rain or the success of a medical procedure.

Developing inquiry skills

Look back at the opening scenario.

Do any of the situations in the tale of Chancer's café involve the Poisson distribution?

Answer: The number of buses arriving at the bus stop.

13.4 Modelling measurements that are distributed randomly

International-mindedness

The Galton board, also known as a quincunx or bean machine, is a device for statistical experiments named after English scientist Sir Francis Galton. It consists of an upright board with evenly spaced nails or pegs driven into its upper half, where the nails are arranged in staggered order, and a lower half divided into a number of evenly-spaced rectangular slots. In the middle of the upper edge, there is a funnel into which balls can be poured. Each time a ball hits one of the nails, it can bounce right or left with the same probability. This process gives rise to a binomial distribution of in the heights of heaps of balls in the lower slots and the shape of a normal or bell curve.

Good simulations and explanations may be found online such as

https://youtu.be/6YDHBFVIvIs

Investigation 8

Conceptual understanding:

Naturally occurring symmetric data where very small and very large data points are relatively rare can be modelled by the normal distribution.

Additional notes

Good quality data sets can be hard to find. Encourage the students bring in data sets from other subjects as well as use the data sets http://mei.org.uk/data-sets (Blackbirds) and https://www.cia.gov/library/publications/the-world-factbook/rankorder/2102rank.html (Life expectancy)

1 No, because it is a set of measurements.

2 Yes

3 Yes

4 **Factual:** Is *E* a continuous random variable?

Answer: Yes, because it represents a set of measurements

5 **Factual:** Does the histogram for *E* show a symmetric bell-shaped curve?

Answer: No

6 **Factual:** Can it be modelled by a normal distribution?

Answer: No

7 The following could be modelled by the normal distribution: a, c, e, f, i, j, l

8 **Conceptual:** In which contexts may you expect the normal distribution to be an appropriate model?

Answer: Naturally occurring symmetric data where very small and very large data points are relatively rare can be modelled by the normal distribution.

Investigation 9

Conceptual understanding:

The parameters of the normal distribution function model the mean and variance in normally distributed data.

Additional notes

The investigation can be enhanced by use of sliders either on the GDC or with dynamic geometry software.

1 **Conceptual:** What does the shape of $f(x)$ tell you about where the probability is distributed most/least densely in the normal distribution?

Answer: The probability is distributed most densely in the middle where the function has its maximum.

2 $\left(\mu, \dfrac{1}{\sigma\sqrt{2\pi}}\right)$

3 $y = 0$

4 **Factual:** Which parameter affects the position of the axis of symmetry of the function?

Answer: μ

5 **Factual:** Which parameter affects the gradient of the function?

Answer: σ

6 **Factual:** In a data set, how do you quantify the central tendency of the data? How do you quantify the spread of the data?

Answer: The central tendency is quantified by the mean and the spread is quantified by the standard deviation or variance.

7 **Factual:** What letters do we use to represent mean and variance?

Answer: μ and σ

8 **Factual:** What is the normal distribution function?

Answer: The normal distribution follows a normal curve – that is, a bell-shaped curve.

9 **Conceptual:** What do the parameters of $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$ model?

Answer: The parameters of the normal distribution function model the mean and variance in normally distributed data.

Investigation 10

Conceptual understanding:

The normal cumulative distribution function predicts probabilities for a given interval of the random variable, and a sketch of the area can support the evaluation of the probability.

Additional notes

Students will ask for clarification of question **7**.

1 7 is the mean of the function, therefore $\mathrm{P}(X \le 7) = 0.5$. Any values larger than 7 will have a probability greater than 0.5.

2 $\mathrm{P}(X \ge 8)$ = 1 – $F(8)$

3 $\mathrm{P}(\boldsymbol{X} \le \mathbf{6})$ = 1 − $F(8)$

4 $\mathrm{P}(\boldsymbol{X} \le \mathbf{6})$ = $F(8)$

5 $\mathrm{P}(6 \le X \le 8)$ = 2 $F(8)$ − 1

6 **Factual:** What is the total area under the function?

Answer: 1

7 They are the same because with continuous random variables the probability of a random variable being **exactly** equal to any given value is 0.

8 0.65, 0.95, 0.997

9 **Factual:** Repeat (8) with your own values of μ and σ. What do you notice?

Answer: The results are the same.

If $X \sim N(\mu, \sigma^2)$, then:	
$P(\mu - \sigma \le X \le \mu + \sigma) =$	0.65
$P(\mu - 2\sigma \le X \le \mu + 2\sigma) =$	0.95
$P(\mu - 3\sigma \le X \le \mu + 3\sigma) =$	0.997

10 **Conceptual:** Given an interval of values of the random variable $X \sim N(\mu, \sigma^2)$, what does the cumulative probability function F quantify?

Answer: The normal cumulative distribution function predicts probabilities for a given interval of the random variable, and a sketch of the area can support the evaluation of the probability.

Developing inquiry skills

Do any of the situations in the café involve the normal distribution?

Answer: The weight of a bag of coffee and the weight of the inspector may follow a normal distribution.

13.6 Distributions of combined random variables

TOK

How well do models, such as the Poisson distribution, fit real-life situations?

Can you use mathematics to describe the world? Consider the use of probability methods in medical studies to assess risk factors for certain diseases.

Do we see mathematics in our world or are we imposing our own mathematical topics onto situations?

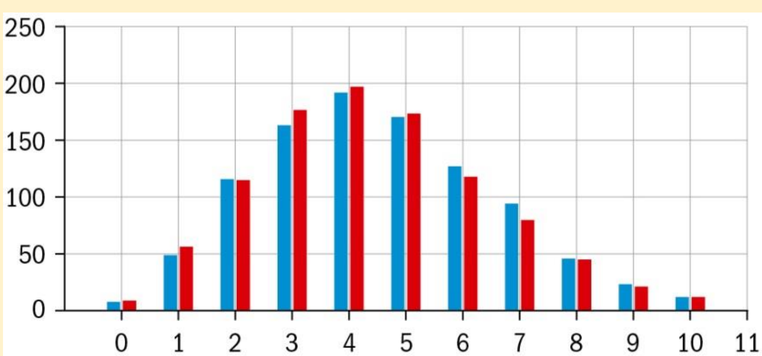
Investigation 11

Conceptual understanding:

A sum of two independently distributed Poisson random variables follows a Poisson distribution.

Additional notes

After the spreadsheet instructions, the students should see two histograms with very similar distributions, such as the following



Further discussion can take place after the investigation regarding the random variable $Z = aX + bY$; why it can only follow the Poisson distribution if $a = b = 1$?

1 **Factual:** What are $E(A + B)$ and $Var(A + B)$?

Answer: $E(A + B) = 4.5 = Var(A + B)$

2 It is true that in the total of the two fields, events are independent and that they occur at a uniform rate so it seems reasonable to state that T should be modelled by a Poisson distribution.

3 **Conceptual:** Given two independently distributed Poisson random variables $X\sim\mathrm{Po}(\lambda)$ and $Y\sim\mathrm{Po}(\alpha)$, what can you predict about the distribution of $Z = X + Y$?

Answer:. $Z\sim Po(\alpha+\lambda)$

A sum of two independently distributed Poisson random variables follows a Poisson distribution.

Investigation 12

Conceptual understanding:

Linear combinations of n independent normal random variables lead to a normal distribution and the mean and variance can be found from the formulae for the mean and variance of linear combinations of n independent random variables.

Additional notes

Use of the GDC to create and add lists of random numbers (of equal length) is well worth applying in this investigation.

Question **6** guides the students to apply the result from **5** inductively, a large step.

1 E(S)=5.5 and Var(S)=2.93

2 Yes.

3 Yes.

4 **Factual:** Is the normal distribution an appropriate model for $S = T + Y$?

Answer: Yes.

5 **Conceptual:** Given two independently distributed normal random variables $T \sim N(\mu_1, {\sigma_1}^2)$ and $Y \sim N(\mu_2, {\sigma_2}^2)$, what can you predict about the distribution of $T = aX + bY$?

Answer: $T = aX + bY$ is distributed normally $T \sim N(a\mu_1 + b\mu_2, a^2{\sigma_1}^2 + b^2{\sigma_2}^2)$

6 **Conceptual:** How can you determine the probability distribution of a linear combination of n independent normal random variables?

Answer: Linear combinations of n independent normal random variables lead to a normal distribution and the mean and variance can be found from the formulae for the mean and variance of linear combinations of n independent random variables.

Investigation 13

Conceptual understanding:

The distribution of sample means of n independent normally distributed random variables allows predictions to be made about the central tendency of a sample.

1 $X_1 + X_2 \sim N(2\mu, 2\sigma^2)$

2 $X_1 + X_2 + X_3 \sim N(3\mu, 3\sigma^2)$

3 $X_1 + X_2 + \cdots + X_n \sim N(n\mu, n\sigma^2)$

4 $\frac{X_1+X_2+\cdots+X_n}{n} \sim \mathrm{N}\left(\frac{n\mu}{n}, \frac{n\sigma^2}{n^2}\right)$

$\Rightarrow \bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$

5 It stays the same.

6 It decreases as n increases.

7 The formula shows that the mean is constant and that the variance is inversely proportional to n.

8 **Factual:** As the size of the aircraft increases, what is the effect on the confidence the pilot has about the average weight of the passengers in the aircraft?

Answer: The confidence increases as n increases.

9 **Factual:** As the size of a sample increases, what happens to the probability that the sample mean will differ from the population mean by a given amount?

Answer: The probability decreases as n increases.

10 **Conceptual:** What does the **distribution of sample means** of n independent normally distributed random variables help you to predict?

Answer: The distribution of sample means of n independent normally distributed random variables allows predictions to be made about the central tendency of a sample.

TOK

Discuss the statement "Without the central limit theorem, there could be no statistics of any value within the human sciences."

You might want to consider that the central limit theorem can be proved mathematically (formalism), but its truth can be confirmed by its applications (empiricism).

Investigation 14

Conceptual understanding:

The central limit theorem allows predictions to be made about the central tendency of a sample even with an unknown distribution of the underlying population.

Additional notes

Use of the GDC to create lists of numbers with different shapes is very useful in this investigation. For example, creating a sequence of 1000 numbers on a GDC gives a data set that can be used in this investigation.

1 **Factual:** In every distribution you create, is the distribution of sample means modelled by a normal distribution when the sample size n is at least 30?

Answer: Yes

2 **Factual:** Do your findings support this statement of the **central limit theorem**

"The mean $\bar{X}$ of a sample of size n taken from **any** population with mean μ and standard deviation σ, can be modelled by a normal distribution with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$ provided that n is at least 30"?

Answer: Yes

3 In the central limit theorem, you do not have information about the type of distribution followed by the population but you do have the mean and variance. With the distribution of sample means, you were given that the population followed a normal distribution and you were given the parameters.

The conclusions are the same, except that there is a need for a large sample size in order to make a good approximation of the distribution of sample means in the central limit theorem.

4 **Conceptual:** What does the **central limit theorem** help you to predict?

Answer: The central limit theorem allows predictions to be made about the central tendency of a sample even with an unknown distribution of the underlying population.

Developing inquiry skills

Look back at the opening scenario. Add a paragraph to the tale of the café that involves the central limit theorem.

Answer: Students' own answers.

Developing inquiry skills

In Chancer's café in the opening scenario, how many examples of probability distributions can you find?

How many of them are discrete? How many are continuous?

How many of these distributions have you learned about in this chapter?

How many of the situations involve combinations of random variables?

Do you need any more information to help you be sure of your choice of model?

How can you name and define precisely the distributions in the Chancer's café scenario that you have not yet learned?

Answer:

- In the opening scenario, how many examples of probability distributions can you find?
    - There are six examples of probability distributions.
- How many of them are discrete? continuous?
    - The day D on which the inspector arrives is a discrete random variable.
    - The length of time T waited at the bus strop is a continuous random variable.
    - The number of buses B arriving at the bus stop is a discrete random variable.
    - The number of failing percolators F is a discrete random variable.
    - The number of trails L needed before finding the correct combination is a discrete random variable.
    - The weight of a bag of coffee E is a continuous random variable.
    - The weight of an inspector W is a continuous random variable.
- How many of these distributions have you learned about in this chapter?
    - B follows a Poisson distribution.
    - F follows a binomial distribution.
    - E and W may follow normal distributions.
- How many of the situations involve combinations of random variables?
    - The total weight of the inspectors is a combination of all their weights.
- Do you need any more information to help you be sure of your choice of model?

   - We know the range of weights of coffee bags but not the shape of the distribution – that information would be needed in order to know if the normal distribution was an appropriate model.
- How can you name and define precisely the distributions in the scenario that you have not yet learned?
   - You can research the uniform distribution (D) and the geometric distribution (L) and determine if they are appropriate models.

Fair game!

Approaches to Learning/learner profile: Collaboration, Communication, Self-management

Exploration Criteria: Presentation (A), Mathematical Communication (B) Personal Engagement (C), Reflection (D) Use of Mathematics (E)

IB Topic: Expected Value, Probability Distributions

Introduction

This task can be great fun for a class or even whole year group to get involved in. It is also possible to invite other classes from other parts of the school where this is viable and it produces great discussions for many different age groups. Sensitivity is obviously required around the topic of gambling and clearly no actual money should change hands! This may be a time-consuming task but given the right occasion (end or beginning of term, special mathematics day, etc.) it can be very rewarding and fun.

The aim is for students to engage in a real-life probability exercise and improve their understandings of probability, probability distributions, expected value and the concept of a fair game. However, there is a lot more to it than that. Students will need to present coherently, be organized, communicate mathematically and reflect on their outcomes. There are also aspects of psychology, advertising, ethics, etc., that may come up in various discussions.

Students should definitely be given an opportunity to reflect on the success of their game and consider what improvements they would make to make this a worthwhile experience. The final 'product' here is two-fold – first it is the game itself, but more important is the write-up or video that explains the game, its success, the mathematics and any improvements.

The task

Put students into pairs or groups of three to design their game.

To start a discussion, you could ask:

What type of game would you play at a carnival, fairground or amusement park?

Students can use dice, spinners, balls, plastic darts, marbles or anything else that they can think of that is available to them that uses probability and chance to win.

When thinking about how to make a profit, you could ask:

What is a fair game?

Is a fair game likely to give you a profit?

A fair game is a game that is not biased toward any player. All players have the same chance of winning.

A fair game is not likely to give a profit.

Each school context for the playing of the games and the resources available will be different. Ensure that students are clear about what is required of them.

There are three main parts to this task:

Part 1 – Design and understand the probabilities involved in your game.

Part 2 – Set up the game in a “class fair”.

Part 3 – Reflect on the success or otherwise of your game.

Part 1 – Design and understand the probabilities involved in your game

Consider the amount of time that will be required for this. There will probably be an expectation that some of this will take place outside of class.

Part 2 – Set up the game in a “class fair”

Think of the available space and how to distribute this space between the groups.

One student from each group must remain at their game to supervise its playing, take the payments and award the prizes.

The other students may visit the other games to play.

Swap the roles regularly so that everyone gets a chance to both supervise their game and play the other games.

Part 3 – Reflect on the success or otherwise of your game

Ensure it is clear that they do not necessarily need to answer all of the questions listed extensively. They are just for guidance.

Extension

Gambling is a controversial topic and will clearly need to be treated differently depending on the context of the school. The Mathematics guide suggests questions that could be considered during this chapter and these are adapted here for this extension. These questions could form the basis of a class debate, a written or videoed response or a blog post, etc. The overlap with Theory of Knowledge is significant but these questions also have some mathematical foundation to them.

What does “the house always wins’ mean? This could be discussed in the context of expected values $E(X) < 0$.

This Quora post has some interesting comments regarding this:
https://www.quora.com/Casinos-Why-does-the-house-always-win

Could mathematics and mathematicians help increase incomes in gambling? Again, students could use the concept of expected values. There is also mathematics that can be discussed around probabilities of particular lottery numbers coming up as there are many misconceptions here and comparing the probabilities in various national lotteries.

How would a mathematician explain luck? Luck has many aspects to it and Mathematics is one of them. The idea that given enough opportunities then some will be lucky (and others will be unlucky) as all results are theoretically possible.

This Quora post is interesting on this
https://www.quora.com/What-is-luck-Is-there-a-mathematical-explanation

14 Testing for validity: Spearman's, hypothesis testing χ^2 test for independence

Essential understandings

Statistics is concerned with the collection, analysis and interpretation of quantitative data and uses the theory of probability to estimate parameters, discover empirical laws, test hypotheses and predict the occurrence of events. Statistical representations and measures allow us to represent data in many different forms to aid interpretation.

Probability enables us to quantify the likelihood of events occurring and so evaluate risk. Both statistics and probability provide important representations which enable us to make predictions, valid comparisons and informed decisions. These fields have power and limitations and should be applied with care and critically questioned, in detail, to differentiate between the theoretical and the empirical/observed. Probability theory allows us to make informed choices, to evaluate risk and to make predictions about seemingly random events.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Correlation and regression are powerful tools for identifying patterns and equivalence of systems.
- Modelling and finding structure in seemingly random events facilitates prediction.
- Different probability distributions provide a representation of the relationship between the theory and reality, allowing us to make predictions about what might happen.
- Statistical literacy involves identifying reliability and validity of samples and whole populations in a closed system.
- A systematic approach to hypothesis testing allows statistical inferences to be tested for validity.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	Investigation
Different correlation coefficients such as Spearman's and Pearson's describe the relationship between two variables for different types of data i.e. ranked and quantitative respectively. Pearson's correlation coefficient is used to test for linearity and Spearman's for a monotonic increase or decrease. Spearman's is used on ranked data.	Investigation 1
A hypothesis test gives a decision rule on whether or not to reject the null hypothesis and the significance level of a test specifies the probability of rejecting a true null hypothesis.	Investigation 2
The normal distribution can be used to test for a population mean, whenever the background population can be modelled by a normal distribution or the sample size large enough for the central limit theorem to apply.	Investigation 3

The expected values give us the average values we would expect, under the assumptions of the null hypothesis, if the sample was selected many times.	Investigation 4
The chi-squared test for independence determines whether a relationship between the two variables exist.	Investigation 5
The chi-squared goodness-of-fit test allows us to determine whether a relationship exists between the data and a distribution such as a normal distribution, binomial distribution or uniform distribution.	Investigation 6
Consideration of type I and type II errors allows a significance level to be chosen that is appropriate to the required degree of risk.	Investigation 7
The *p*-value reflects the strength of evidence against an assumed true null hypothesis and does not predict the likelihood of a true null hypothesis.	Investigation 8

Syllabus sections covered in this chapter:

- SL4.11
- SL4.10
- AHL4.12
- AHL4.14
- AHL4.16
- AHL4.18

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 624	Page 631: Example 2 Page 654: Example 11 Page 679: Example 19	Page 626: Example 1 Page 631: Example 2 Page 634: Example 3 Page 636: Example 4 Page 640: Example 5 Page 641: Example 6 Page 644: Example 7 Page 646: Example 8 Page 647: Example 9 Page 648: Example 10 Page 654: Example 11 Page 658: Example 12 Page 661: Example 13 Page 662: Example 14 Page 666: Example 15 Page 667: Example 16 Page 678: Example 18 Page 679: Example 19	Pages 627, 637, 649, 655, 668, 682

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 683	Page 684	N/A

Developing inquiry skills

This type of problem may be one that is familiar to many students from field work in their group 3 or 4 subject, but it will ask the students to consider the problem in more depth than they will have experienced before.

At the start of the chapter they are asked to review some of the techniques from earlier chapters, including the importance of selecting a random sample. Points to consider here include how do you divide the forests into smaller sections, do you take more samples from the more densely packed sections or does each section have equal chance of selection? If a small section is selected at random how do you select the tree from within that section? How might you try to guarantee that all the trees are of an equal age – and would it matter if not?

The statistics they have met previously are the mean, standard deviation, median, quartiles and inter-quartile range.

They are also introduced to inferential statistics and are asked whether the statistics taken from the sample provide enough evidence to answer the question posed?

The extra information they might suggest is that it would be useful to know the probability of getting two such different samples if there was in fact no difference in average height of the trees.

Though they are not asked to do it, a box and whisker plot would be an excellent way to compare the two samples and provide some idea about whether or not the hypothesis is true.

Write down any similar inquiry questions you might ask if you were asked to look at other sets of data to decide whether a statement about them was true or not. It might be, for example, the heights of children, or the sizes of pebbles on a beach, or the fuel consumption of cars. What would you need to think about in each case?

> **Answer:** Many of the questions would be similar. How might you collect a sample that truly reflects the distribution in the population? In the case of children, it needs to be decided are you comparing children or boys and girls, are you comparing them across ages or all the same age? How large a sample would you need to take before you can be reasonably sure it represents the population?
>
> In all the examples you might be justified in assuming they are normally distributed. But how might you test that? What can you do if they are not normally distributed, can you make any assumptions? If testing for the mean how might you measure how far apart the means of the two samples are before you say there is sufficient evidence that the two populations have different means?

14.1 Spearman's rank correlation coefficient

Investigation 1

Conceptual understanding:

Different correlation coefficients such as Spearman's and Pearson's describe the relationship between two variables for different types of data i.e. ranked and quantitative respectively.

Pearson's correlation coefficient is used to test for linearity and Spearman's for a monotonic increase or decrease. Spearman's is used on ranked data.

Additional notes:

This investigation is to help the students understand how to calculate Spearman's rank correlation and what is meant by values close to 1 or −1

1 Using a GDC gives the value $r = 0.956$.

Comments should be that the PMCC is close to 1 which implies that the data can be reasonably modelled by a linear model. From the graph though it looks like a logistic model would be more suitable (or simply it looks like the rate of increase is not constant).

2 **a**

	A	B	C	D	E	F	G	H
***X* rank**	3	8	7	5	6	2	4	1
***Y* rank**	3	8	7	5	6	2	4	1

b The value is 1.

c Because the ranks are matched exactly the PMCC is equal to 1. The feature that leads to this result is that the points on the graph are always increasing.

3 **a** $r = -0.849$. The PMCC is not as strong but indicates the data can be approximated by a negative linear model. The graph indicates that there is a sudden change in the model between $T = 32.3$ °C and $T = 36.4$ °C.

b

	A	B	C	D	E	F	G	H
***T* rank**	8	7	6	5	4	3	2	1
***Y* rank**	2	1	3	4	5	6	7	8

c $r = -0.976$. This is very close to -1 indicating that generally one variable decreases as the other increases.

d The only exception is point B. Point H being a long way from the others did not greatly affect the data as it was done on the ranks.

4 **Factual:** What type of data is used for Spearman's?

Answer: Any data that can be ranked is suitable to calculate Spearman's rank correlation coefficient, which is the PMCC of the ranks of the data.

5 **Factual:** What type of data is used for Pearson's?

Answer: Only quantitative data can be used for the PMCC.

6 **Conceptual:** What do these correlation coefficients tell you about the relationship between two variables? When do you use which?

Answer: Different correlation coefficients such as Spearman's and Pearson's describe the relationship between two variables for different types of data i.e. ranked and quantitative respectively.

Pearson's correlation coefficient tests for linearity and Spearman's for a monotonic increase or decrease on ranked data.

International-mindedness

In 1956, Australian statistician, Oliver Lancaster made the first convincing case for a link between exposure to sunlight and skin cancer using statistical tools including correlation and regression.

The correlation between smoking and lung cancer was "discovered" using mathematics.

Science had to justify the cause

TOK

What is the difference between correlation and causation?

To what extent do these different processes affect the validity of the knowledge obtained?

Correlation is the idea of modelling a pattern based on data. Causation is using data as proof that one thing causes the other.

Does correlation need causation?

What is a cause and effect relationship?

Is making a model for a given situation valid as personal knowledge?

Reflect: How does a scatter graph help you to interpret and compare two data sets?

Answer: it can indicate if a linear relationship is likely and can help identify any outliers

When is it better to use Pearson's correlation coefficient and when is it better to use Spearman's?

Answer: Pearson's is used to test for linearity and Spearman's for a monotonic increase or decrease. Spearman's can be used on ranked data and is affected less by outliers.

14.2 Hypothesis testing for the binomial probability, the Poisson mean and the product moment correlation coefficient

Initial questions at the start of the section

What does this tell us about the mean or standard deviation of all the plants in the field?

What might the accuracy of the prediction depend on?

Answer: The probable answers are that the students would expect the mean and standard deviation of the sample to be close to those of the population. They might be able remembering that the expected value of the sample mean is the population and then argue by analogy that they would expect the same to be true of the standard deviation.

The accuracy will depend on the variance of the population and the sample size. Again they might remember calculating the variance of the sample mean in the previous chapter but at this stage qualitative answers will be all that is required

Investigation 2

Conceptual understanding:

A hypothesis test gives a decision rule on whether or not to reject the null hypothesis and the significance level of a test specifies the probability of rejecting a true null hypothesis.

Additional notes:

This investigation introduces the idea of hypothesis testing and a significance level through having the students devise their own critical region for a test. The intuitive idea of unlikely outcomes leading to a rejection of the null hypothesis is developed more formally as the chapter develops.

1 **a** For H_0: $p \neq 0.5$ you do not know which probability to take to do your test.

b H_1: $p \neq 0.5$

2 **a** Answers here will vary but a sensible estimate would be 0, 1, 9, 10 for rejecting the null hypothesis.

b Again answers might vary, and might include that there are no values. Otherwise it is likely that 2 or 8 heads might lead a student to be suspicious of the coin. The best action in this case would be to toss the coin a few more times or, equivalently, repeat the test.

An extra question, not asked here, might be would they be more suspicious of 8 heads out of 10 or 16 heads out of 20, or would both be the same. This could lead to a discussion about the importance of having a large sample. This aspect will be taken up later in the chapter.

3 **a** Likely significance levels are:

0, 10 Heads	0.00195 or 0.195%
0,1,9,10 Heads	0.0215 or 2.15%
0,1,2,8,9,10	0.108 or 10.9%

b Usual responses would be yes for the first two and no for the last one, which would lead to the general use of 5%.

4 **Conceptual:** What is meant by a significance level of a test?

Answer: The significance level of a test is the probability of a test statistic being in the critical region if the null hypothesis is true, hence it is the probability of rejecting the null hypothesis when it is true. It is normally given as a percentage.

5 **a** This is the probability of rejecting the null hypothesis that the coin is fair if it is fair.

b The students need to choose a different critical region and evaluate the probability of being in it if the coin is fair. This can then be converted into a percentage.

c Again this leads to a discussion about what the right value might be, and possibly the choice of 5% as a limit.

d Though they don't have to be you would probably need a reason for favouring heads over tails or vice versa.

6 **a** It is always possible to get a long run of heads even when the coin is fair so you can never be sure. But as the probability of an event occurring by chance under the null hypothesis becomes increasingly small then the more likely it is that the alternative hypothesis is true if the result of the test falls in the critical region.

b No, it might still provide evidence for the null hypothesis being false (for example if the *p*-value is between 0.05 and 0.10), but not enough to reject the null hypothesis. In this case further testing would usually be recommended.

6 **Conceptual:** What information do we obtain from a hypothesis test.

Answer: A hypothesis test gives a decision rule on whether or not to reject the null hypothesis and the significance level of a test specifies the probability of rejecting a true null hypothesis.

TOK

If the result of a test is significant, what do we actually know?

Some research journals have "banned" *p*-values from their articles because they deem them too misleading as to the certainty of the results https://www.sciencenews.org/blog/context/p-value-ban-small-step-journal-giant-leap-science

Reflect: How do you find a *p*-value for the binomial probability?

How do you find the critical region for the binomial probability?

Answer: The answer is illustrated in Example 2.

Reflect: When is it appropriate to test for a binomial probability?

Answer: Tests for parameters of distributions should only be used if the conditions for the random variable to be from that distribution are satisfied.

In the case of the binomial distribution this includes their being repeated, identical trials with constant probability of success and for which the results are independent of the other trials.

Reflect: How do you find a *p*-value for the mean of a Poisson?

How do you find the critical region for the mean of a Poisson?

Answer: The answer is illustrated in Example 3.

Reflect: When is it appropriate to test for a Poisson mean?

Answer: Tests for parameters of distributions should only be used if the conditions for the random variable to be from that distribution are satisfied.

In the case of the test for a Poisson mean:

1. Events are independent.
2. Events occur at a uniform average rate (during the period of interest).

Reflect: How do you find a *p*-value when testing for PMCC = 0?

Answer: The answer is given in Example 4 and the key point box below it.

Reflect: When is it appropriate to apply linear regression to bivariate data?

Answer: When the shape of the graph indicates that the data is approximately linear.

14.3 Testing for the mean of a normal distribution

Investigation 3

Conceptual understanding:

The normal distribution can be used to test for a population mean, whenever the background population can be modelled by a normal distribution or the sample size large enough for the central limit theorem to apply.

The distribution $\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ gives the critical region when hypothesis testing with the normal distribution.

Additional notes

This investigation considers the test for a population mean and in particular the calculation of the critical region. This is necessary in order to create decision rules and for working out type II errors later in the chapter.

1 In a stable population you would expect roughly equal numbers for each age group with a gradual reduction in numbers as the ages rise followed a fairly quick decline for the later ages.

2 With such a large sample the central limit theorem will apply.

3 H_0: $\mu = 40.0$, H_1: $\mu > 40.0$

4 As n increases the variance decreases which means it is more likely that $\bar{X}$ will be close to μ. Students need to be aware that roughly 95% of all data selected from a population with mean μ is likely to be within 2 standard deviations of the mean.

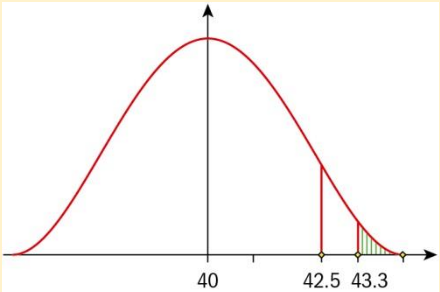
5 **a** Distribution is $\mathrm{N}\left(40, \frac{20.2^2}{100}\right)$

b $\mathrm{P}(\bar{X} < \boldsymbol{a}) = \mathbf{0.95} \Rightarrow a = \boldsymbol{invnorm}\left(0.95, 40.0, \frac{20.2^2}{100}\right)$.

Different calculators will find this value in different ways.

Critical region is $\bar{X} > 43.3$

6 42.3 < 43.3 not significant, so insufficient evidence to reject H_0: $\mu = 40.0$.



7 $\mathrm{P}(\bar{X} > 42.3) = 0.127 > 0.05$ so also not significant.

The p-value is the probability of getting the result obtained or a more extreme one.

8 **Factual:** How do you find the critical value of a one-tailed test?

Answer: The critical value is a such that $\mathrm{P}(\bar{X} > a)$ is equal to the significance level, or $P(\bar{X} < a)$, depending on the direction of the tail.

9 **Conceptual:**

a What conditions are necessary to be able to use the normal distribution to test for a population mean?

b If the conditions in **a** apply which distribution is used to find the critical region for a test for a population mean?

Answer: The normal distribution is used to test for a population mean, whenever the background population can be modelled by a normal distribution or the sample size large enough for the central limit theorem to apply.

The distribution $\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ is used to find the critical region for hypothesis testing with the normal distribution.

TOK

When is the normal distribution a valid model?

Can we use the scientific method from the natural sciences to conduct a normal probability exercise?

When can we use when can we not use a normal distribution?

Normal distribution assumptions are important to note because so many experiments rely on assuming a distribution to be normal. In most cases, is the assumption of normality a reasonable one to make?

Reflect: What are two-tailed tests?

Answer: Two-tailed tests have as their alternative hypothesis that the population mean is not equal to a given value. The null hypothesis is therefore rejected if the sample mean is either too big or too small

Reflect: When should the *t*-test be used instead of the *z*-test?

Answer: When the population variance is unknown.

How do you calculate the degrees of freedom for a *t*-test?

Answer: The number of elements in the sample, minus 1.

How do you find the unbiased estimator for the variance if given the sample variance?

Answer: The unbiased estimator can be calculated from the standard deviation using the formula

$$s_{n-1}^{s} = \frac{n}{n-1} s_n^2$$

TOK

In the absence of knowing the value of a parameter, will an unbiased estimator always be better than a biased one?

It is usual for an unbiased statistic to be preferred over a biased statistic, because there is a tendency for the biased statistic to under or overestimate the actual value of the parameter. Some questions that you might like to pose for students include:

- Is it always a case of pass or fail or are there varying levels?
- What if the tests look for different features?
- You might ask students:
- Who would prefer the results of a biased estimator?
- What if you knew what you wanted the data to say?
- Could there be people who would strongly prefer biased estimators, provided the bias were in their favor?

Reflect: What is a confidence interval?

Answer: A confidence interval is a range of values between which we are 95% (for example) certain the population mean will lie – more formally on 95% of the occasions such data would be produced the population mean would be within the confidence interval.

Reflect: What is the effect of sample size on the width of the confidence interval?

Answer: As the sample size increases the width of the confidence interval decreases.

Reflect: What is the difference between pooled and non-pooled and which should you use?

Answer: For a pooled sample you are assuming both populations have the same variance, for non-pooled you are not. In exams you should use the pooled sample.

TOK

Can we claim that one product is better on average than another if there is a large overlap between the confidence intervals of the two means?

How can statistical data influence our decision making in this case?

It is common to hear people say that the confidence intervals of the two groups overlap, hence the difference is not statistically significant.

This statement is incorrect. Overlapping confidence intervals say nothing about statistical significance, but it is a common mistake to infer a lack of statistical significance, mainly due to seeing the opposite statement as non-overlapping confidence intervals – imply significance.

You might ask students to interpret the saying "a little bit of knowledge is a dangerous thing".

Reflect: How is a paired sample *t*-test different from a two-sample *t*-test?

Answer: In a paired sample *t*-test there is an obvious pairing between the element of the first dataset and the elements of the second.

What do you need to do before testing a paired sample?

Answer: Before doing a paired sample *t*-test you need to subtract the elements in one of the data sets from the elements in another.

14.4 χ^2 test for independence

Investigation 4

Conceptual understanding:

The expected values give us the average values we would expect, under the assumptions of the null hypothesis, if the sample was selected many times.

Additional notes:

The aim of this investigation is to develop the idea of expected values under the null hypothesis of independence and to explore possible test statistics to measure the difference between observed and expected values.

This is the table of **observed** frequencies, f_o, and is called a **contingency table**.

	Tennis	Badminton	Squash	Totals
Male	10	7	16	33
Female	9	13	5	27
Totals	19	20	21	60

1 The probability that a person chosen at random is male is 33/60.

2 The probability that a person chosen at random likes tennis best is 19/60.

3 If these two probabilities are independent then the two probabilities can be multiplied

$\frac{33}{60} \times \frac{19}{60} = 0.174$

4 There are 60 people in total so to find the expected number of males who like tennis best we multiply by 60

$\frac{33}{60} \times \frac{19}{60} \times 60 = 10.45$

5

	Tennis	Badminton	Squash	Totals
Male	10.45	11	11.55	33
Female	8.55	9	9.45	27
Totals	19	20	21	60

Note: expected values must be greater than 5. If there are expected values less than 5 then you will need to combine rows or columns. This is demonstrated in Example 11, and can be discussed when the chi-squared statistic is explained later in this investigation.

	Tennis	Badminton	Squash	Totals
Male				33
Female				27
Totals	19	20	21	60

6 It can easily be shown that any two entries are sufficient as the rest can be calculated by subtracting from the totals.

7 If the table had three rows and three columns, then four entries would be sufficient as the third entry in each row and column could be deduced from the totals.

8 Similarly if the table had three rows and four columns, the smallest number of entries that you need to would be 2 × 3 = 6.

9 If the table had n rows and m columns then the smallest number of entries would be $(n-1)(m-1)$.

10 Factual: What does the number of degrees of freedom represent?

Answer: The number of free choices of expected values, given the restrictions on the totals, is called the degree of freedom.

11 Conceptual: What do the "expected values" tell us?

Answer: The expected values give us the average values we would expect, under the assumptions of the null hypothesis, if the sample was selected many times.

Investigation 5

Conceptual understanding:

The chi-squared test for independence determines whether a relationship between the two variables exist.

1 The entries for observed and expected values in the first column are both 0.45 apart. Both entries in the second column are 4 apart and the two entries in the third column are 4.45 apart.

In absolute terms therefore the entries in the third column have the greatest gap between observed and expected values.

2 They sum to zero as positive and negative terms cancel each other out, so not suitable in measuring distance

3 Squaring the distances before adding them would eliminate the minus signs so the differences would no longer cancel out.

4 A disadvantage is that it will give extra weight to the larger differences, for example in the tables above a difference of 0.45 will become smaller when squared, while a difference of 4.5 will become much larger.

Also a difference of 3 for example between 1 and 4 indicates a big change, a difference of 3 between 242 and 245 is only a small change but under this measure they both count the same.

5 $\dfrac{(10-10.45)^2}{10.45} + \ldots = 7.09$

6 It will be larger as the χ^2 statistic will reach it less frequently.

7 7.09 > 5.991 which is significant so reject the null hypothesis and hence preferred racket game is not independent of gender.

8 Calculators will do the test in different ways but all should give the test statistic of 7.09.

Most calculators will not give the expected values on the summary page so students will need to know how to find them to check that they are all greater than 5.

9 From the GDC, the p-value is 0.0289

$0.0289 < 0.05$, so the null hypothesis is rejected.

So, preferred racket game is not independent of gender.

10 **Factual:** What are the null and alternative hypotheses for a chi-squared test for independence?

Answer:

H_0: the two variables are independent of each other.

H_1: the two variables are not independent of each other.

11 **Conceptual:** How can you determine if there is a relationship between two variables?

Answer: The chi-squared test for independence determines whether a relationship between the two variables exist.

More particularly:

- If the chi-squared test statistic is less than the critical value then you accept the null hypothesis and if it is greater than the critical value then you do not accept it.
- If the p-value is greater than the significance level (0.01, 0.05 or 0.10) then you accept the null hypothesis and if it is less than the significance level you do not accept it.

- You can either use the test statistic and the critical value or the *p*-value and the significance level to reach a conclusion to the test.

Reflect: When do you need to combine columns/rows?

Answer: When any of the expected values are less than 5

Developing inquiry skills

Let the height of the trees in the opening problem be h. Divide the heights into small, medium and large where small is $h \leq 4.5$ m, medium is $4.5 < h \leq 5$ and large $h > 5.0$.

Use these categories to form a contingency table for the heights in areas A and B and test at the 5% significance level whether the height of a tree is independent of the area in which it is growing.

Does the conclusion of the test support the hypothesis that the trees from area A, on average, have a greater height than those from area B.

Justify your answer.

Answer: The contingency table for the test is

	Section A	Section B
Small	3	9
Medium	7	9
Large	14	6

All the expected values are greater than 5.

As always the test should state the hypotheses and as the critical value is not given they should use the *p*-value.

H_0: The height of a tree is independent of the section of the area in which it is growing.

H_1: The height of a tree is not independent of the section of the area in which it is growing.

p-value = 0.0398

This is significant at the 5% level so the null hypothesis is rejected and there is strong evidence that the size of the pebbles is not independent of the area in which it is growing.

This does support the hypothesis that the trees in section A are larger than those in section B.

It does not, though, allow us to answer the hypothesis as simply knowing the heights are not independent of area does not tell us in what way they are different.

TOK

What does it mean if a data set passes one test but fails another?

Some questions that you might like to pose for students include:

Is it always a case of pass or fail or are there varying levels?

What if the tests look for different features?

How certain is the mathematical knowledge gained from one test?

Does the ability to test only certain parameters in a population affect the way knowledge claims in the human sciences are valued?

14.5 χ^2 goodness-of-fit test

Investigation 6

Conceptual understanding:

The chi-squared goodness-of-fit test allows us to determine whether a relationship exists between the data and a distribution such as a normal distribution, binomial distribution or uniform distribution.

Additional notes:

This investigation introduces the chi-squared goodness-of-fit test and draws out its links with the test for independence.

1 $\frac{1}{6}$

2 $300 \times \frac{1}{6} = 50$

3

Number	Expected frequency
1	50
2	50
3	50
4	50
5	50
6	50

4 **Factual:** Is the formula for the χ^2 test suitable to test whether Jiang's results fit a normal distribution?

Answer: Yes, because it measures the distance observed values are from expected values.

5 H_1: Jiang's results do not satisfy a uniform distribution or the die is not fair.

6 Though this is unlikely to be required in examinations use of the formula will help understanding of how a test statistic relates to critical values.

χ^2_{calc}=22.24

22.24 > 11.07, so reject the null hypothesis that the data comes from a uniform distribution and hence the die is unlikely to be fair.

7 Because the total for the expected values has to match the total for the observed values there are $n - 1$ degrees of freedom where n is the number of cells.

8 6 – 1 = 5 degrees of freedom.

9 p-value = 4.71×10^{-4}.

10 $4.71 \times 10^{-4} < 0.05$, so reject the null hypothesis that the data comes from a uniform distribution and hence the die is unlikely to be fair.

11 Conceptual: What is the purpose of the chi-squared goodness-of-fit test?

Answer: The chi-squared goodness of fit test tells us whether the data obtained is likely to have been drawn from a given distribution

Reflect: How do you perform a goodness of fit test for a uniform or normal distribution?

Answer: The answers are given in Investigation 5 and Example 12.

Reflect: How do you perform a goodness-of-fit test for a binomial or Poisson

distribution?

Answer: The answers are given in Examples 13 and 14.

TOK

In practical terms, is saying that a result is significant the same as saying that it is true?

How does language influence our perception?

How do the words that we use, like significant and true, play a part in our analysis if taken out of context? Can words that we use in other areas give numbers a different visualization that even students not enamoured with mathematics will likely find distinctive and stimulating?

Developing inquiry skills

	Section A	Section B
Small $h \leq 4.5$	3	9
Medium $4.5 < h \leq 5.0$	7	9
Large $h > 5.0$	14	6

From previous research it is known that the heights of this species of tree follow a normal distribution with a mean of 4.9 m and a standard deviation of 0.5 m.

Use the data above to test the heights of the trees from each of the areas separately and see if the observed values are consistent with both samples being taken from this distribution.

Is there sufficient evidence to say these two samples were not taken from the given normal population?

What do your results suggest about the likelihood of the trees from area A having a greater height than those from area B?

Answer:

Section A

	Observed	Expected
Small $h \leq 4.5$	3	5.08
Medium $4.5 < h \leq 5.0$	7	8.82
Large $h > 5.0$	14	10.1

p-value = 0.255 not significant so no evidence to reject the null hypothesis that they come from a $N(4.9, 0.5^2)$ population.

Section B

	Observed	Expected
Small $h \leq 4.5$	9	5.08
Medium $4.5 < h \leq 5.0$	9	8.82
Large $h > 5.0$	6	10.1

p-value = 0.0957 not significant so no evidence to reject the null hypothesis that they come from a $N(4.9, 0.5^2)$ population.

Hence though these are relatively small values it is still reasonable to assume the pebbles come from a normal distribution and hence the *t*-test can be performed in the next section.

When the samples are combined the *p*-value is 0.786. Which does seem to indicate that the trees are clusters from the same parent population.

Reflect: How are the degrees of freedom calculated if extra parameters are estimated?

Answer: The number of degrees of freedom is the number of cells minus one, and minus one for each parameter estimated.

14.6 Choice, validity and interpretation of tests

TOK

Do you think that people from very different backgrounds are able to follow mathematical arguments, as they possess deductive ability?

You might want to use this TOK session to begin a discussion about whether all people possess the same reasoning ability at the same level. Some might argue that our ability to reason distinguishes us from the rest of the animal kingdom. You might want to ask where students have used reason today. As a counterclaim you could point to people who have a diminished capacity for reasoning, such as the mentally disabled, and ask if they are not human. An attention-grabbing debate is sure to follow.

Reflect: What is a systematic error?

Answer: A systematic error will have a non-zero mean, and often follow some kind of pattern,

What are the advantages and disadvantages of using a questionnaire to collect data?

Answer: A multiple choice or short answer survey will provide data that is easy to analyse but it needs to be used carefully for a variety of reasons.

It will lose a lot of information, subjects might have very different reasons for ticking the same box.

People might not answer honestly (particularly if the survey is not anonymous)

The question might be interpreted in a different way to that intended by the compiler of the questionnaire – the compiler needs to ensure it means the same to different age groups, language speakers, cultures etc.

The question might not actually provide the data needed for the research. For example a question might ask about nationality when the compiler was more interested in which country the subject lived.

The questionnaire needs to be complete, as extra questions cannot be asked later.

Why is important to collect appropriate data for a test?

Answer: For a test to be valid appropriate data needs to be used.

TOK

Given that a set of data may be approximately fitted by a range of curves, where would we seek for knowledge of which equation is the "true" model?

Is it preferable to use sense perception when examining data or is reasoning with figures a better strategy?

Statistics forms the backbone of data science and a sound knowledge of statistics can help to make better decisions. On one hand, descriptive statistics helps us to understand the data and its properties by use of central tendency and variability. On the other hand, inferential statistics helps us to infer properties of the population from a given sample of data. "Knowledge of both descriptive and inferential statistics is essential for an aspiring data scientist or analyst." – D. Gupta.

TOK

When is it more important not to make a type I error and when is it more important not to make a type II error?

Pose the question – Is there one type of error that's more important to control than another?

Break the class into three groups.

Have one group argue for type I and another for type II.

They present their findings to group 3 who decide, with reasons, when it is more important not to make a type I error and when it is more important not to make a type II error?

Investigation 7

Conceptual understanding:

Consideration of type I and type II errors allows a significance level to be chosen that is appropriate to the required degree of risk.

Additional notes:

This investigation explores the nature of type I and type II errors

1 H_0: $\mu = 0.856$, H_1: $\mu > 0.856$

2 $P(\bar{X} > a) = 0.05 \Rightarrow P(\bar{X} < a) = 0.95$

$a = \text{invnorm}\left(0.95, 0.856, \frac{0.01}{\sqrt{40}}\right) = 0.8586$

Critical region is $\bar{X} > 0.8586$. The null hypothesis will be rejected if the sample mean is greater than 0.8586 and the alternative that the new drug is better will be accepted.

3 0.05

4 a = 0.8586

5 Because you need to work out the probability of rejecting H_0 given H_1 is true and so would need a specific value for the mean in the calculation.

6 Probability of a type II error = $P(\bar{X} < 0.8586 | \mu = 0.858) = 0.647$.

This is very large when compared with the probability of a type I error, indicating rejecting a new drug when it is better is less important than accepting a new one when it is not.

7 **a** $P(\bar{X} > a) = 0.01 \Rightarrow P(\bar{X} < a) = 0.99$

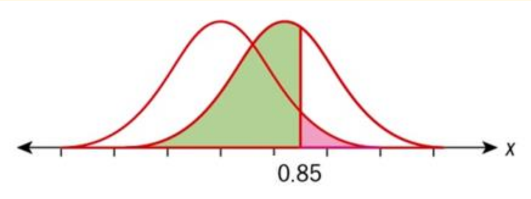
$a = \text{invnorm}\left(0.99, 0.856, \frac{0.01}{\sqrt{40}}\right) = 0.8597$

Critical region is $\bar{X} > 0.8597$.

b $P(\bar{X} < 0.8597 | \mu = 0.858) = 0.8589$.

c Decreasing the probability of a type I error increases the probability of a type II error.

8



9 **a** There are several online sites that allow students to vary different values and to see the effect this have on type I and II errors. They should notice how the boundaries for both errors are always in line and that as the probability of one increases the probability of the other decreases.

As the sample size varies for a fixed significance level the probability of a type II error decreases. So increasing the sample size might allow both errors to be reduced.

b As the sample size increases the standard deviation decreases, as it is equal to $\frac{\sigma}{\sqrt{n}}$, so the sample mean is increasingly likely to be close to the true population mean, making errors less likely.

10 **Factual:** Explain what is meant by a type I and type II error and why is it important to considering both.

Answer: A type I error is rejecting H_0 when it is true and a type II error is accepting H_0 when it is not true.

It is important to consider both because as one decreases the other increases, so the tester needs to decide which is the more important error not to make.

11 **a** p-value = 0.0368 < 0.05 hence the result is significant at the 5% level.

b The result has been shown to be statistically significant, but an increase from 0.856 to 0.8568 might not be sufficient to lead to any improved benefit for the patients. With large samples a statistically significant improvement can come from a very small actual improvement.

12 **Conceptual:** Why is it important to consider type I and type II errors?

Answer: Consideration of type I and type II errors allows a significance level to be chosen that is appropriate to the required degree of risk.

Investigation 8

Conceptual understanding:

The *p*-value reflects the strength of evidence against an assumed true null hypothesis and does not predict the likelihood of a true null hypothesis.

Additional notes:

This investigation is not explicitly on the syllabus but it deals directly with the interpretation of statistical values and combines many aspects of the course.

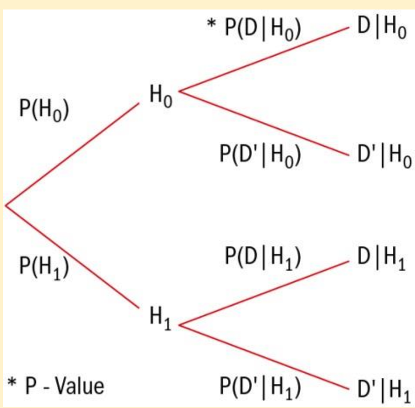
The misinterpretation of the *p*-value is a common one in statistical literature. One of the problems was touched on in question 11 from Investigation 7 when the difference between statistically significant and clinically (or 'actually') significant was discussed.

This section indicates how, if extra information is available, conditional probability can be used to find the probabilities of either the null or alternative hypotheses being true, which is often the more interesting question.

1 **a** $\mathrm{P}(H_0|D)$

b $\mathrm{P}(D'|H_0)$

2 **a** and **b**



c **i** The formula comes directly from Bayes theorem or from the conditional probability formula $\mathrm{P(H_0}|D) = \frac{\mathrm{P(H_0)\times P(}D\mathrm{|H_0)}}{\mathrm{P(}D)}$ and the tree diagram.

ii $\mathrm{P(H_1}|D) = \frac{\mathrm{P(H_1)P(}D\mathrm{|H_1)}}{\mathrm{P(}D)} = \frac{\mathrm{P(H_1)\times P(}D\mathrm{|H_1)}}{\mathrm{P(H_1)\times P(}D\mathrm{|H_1)+P(H_1)\times P(}D'\mathrm{|H_1)}}$

3 **a** **i** $\mathrm{P}(\bar{X} > 15.52|\mu = 10) = 0.0499$

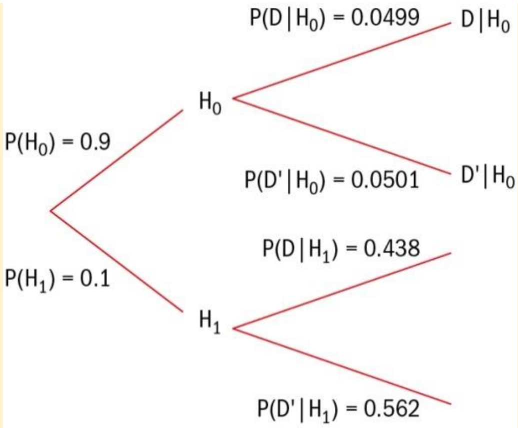
ii 0.95 is the probability that $\mathrm{P}(\bar{X} < 15.52|\mu = 10)$ not $\mathrm{P}(\mu > 10|\bar{X} > 15.52)$.

iii There are many possible answers to this and opinions can range quite widely.

A question that can be asked is "how does the answer to **3 ii** affect their decision"?

b $\mathrm{P}(\bar{X} > 15.52|\mu = 15) = 0.438$

c **i**



ii 0.9 × 0.0499 + 0.1 × 0.438 = 0.0887

iii $\mathrm{P}(\mathrm{H}_1|D) = \frac{0.1\times0.438}{0.0887} = 0.494$

iv There is no strong evidence that the camp is using the illegal treatment but further tests would be needed in order to make sure.

5 Conceptual: What does a *p*-value actually tell you? What is a common misconception?

Answer: The *p*-value reflects the strength of evidence against an assumed true null hypothesis and does not predict the likelihood of a true null hypothesis.

Developing inquiry skills

The initial claim was: the mean height of the trees from area A is smaller than the mean height of the trees from area B.

Which would be the best test to use to address this claim?

What conditions are necessary and have you tested to see if these conditions have been met?

Carry out the test and state the conclusion.

Answer: Being able to choose the appropriate test is an important part of the statistics section of the course. In this case the best test would be the two-sample *t*-test.

The sample size is less than 30 but the chi-squared test suggests it is reasonable to assume a normal distribution. Using the given mean the test result was not significant and hence using the means of each sample as an estimate of their population mean, which would be the more usual test, would certainly not be significant.

H_0: The mean heights of trees from section A is equal to the mean height of trees from section B

H_1: The mean heights of trees from section A is smaller than the mean heights of trees from section B

This is a one-tailed *t*-test.

p-value = 0.0961

Not significant so insufficient evidence at the 5% level to reject H_0 that the mean diameters are equal.

Rank my maths!

Approaches to Learning/learner profile: Collaboration, Communication

Exploration Criteria: Personal engagement (C), Reflection (D) Use of mathematics (E)

IB Topic: Spearman's rank, hypothesis testing

Introduction

Statistics lends itself to explorations very well. In this task students are given an opportunity to design and run an experiment that looks at students' rankings of something and compares these. Spearman's rank can be used in the internal assessment as it gives students the opportunity to statistically analyse data that they collect on their own which lends itself to good opportunities for personal engagement (Criterion C) and Reflection (Criterion D). By extending the problem and demonstrating a good understanding of hypothesis testing students could also be using a sophisticated level of mathematics.

This task is based around Spearman's rank although any of the topics in this chapter could potentially lend themselves to an "experiment" of this kind – students are asked to consider this in the extension task at the end.

The task

For this task, arrange students into groups of three if possible (groups of four would also work, but would mean more results to analyse). Given the nature of the task, it may be worth students being randomized (although not randomized and allowing students to choose groups also provides interesting discussions).

The experiment

Preferably students will be working in groups of three, so in each group there will be one experimenter and two students who are the subject of the experiment.

The experimenter is going to determine whether there is any similarity between the tastes of other people in their group by ranking a set of something from the least to most favorite.

The questions should lead to some interesting discussions that will vary depending on the make-up of groups (for example: how good friends they are, what shared experiences they have had, etc.) and the choice of what is going to be ranked.

For example: students could use Spearman's rank to compare music tastes. The experimenter could choose a few (say between six and ten for ease) different pieces of music, for example, and compare the rankings of these between the other students in their group.

Students could also think about designing a similar experiment to compare ranks in students' taste for films, art, hobbies, food, etc.

In the case of the music example the experimenter could, for example:

- choose a random playlist or the first few random songs in their music collection
- deliberately choose particular songs of a particular genre or a mix of songs from across genres.

The decision about which songs are chosen will change the hypothesis being tested slightly.

You could ask:

In what way will the hypothesis be affected by the choice of music?

The hypothesis could change, for example, between "music tastes are the same/different" and "taste in songs within a particular genre are the same/different".

For their selection, the experimenter could use, for example: film clips, art work, list of actors, list of hobbies, pictures of celebrities, etc.

Students should be encouraged to discuss what it would be ethical and unethical to test.

The prediction will be revisited and reflected on later and so is an important step.

The experimenter should not give any indication of bias or preference themselves when providing instructions.

Here is an example of a table that the experimenter could use:

Item (e.g. song) number	**Student 1 rank**	**Student 2 rank**
1		
2		
3 ...		

It is important that the other students don't collaborate or communicate so that students do not come to an agreement/disagreement, but they choose independently.

The results

If there are four students in a group, then they will need to repeat the experiment for first and third and then second and third students.

Students should consider both size and direction of the Spearman rank values.

The statistical test

What spearman's rank value would mathematically be considered to be high?

It is possible to test the significance of a relationship between the two rankings. The test is similar to the one conducted for the product moment correlation coefficient.

Conduct a hypothesis test for the Spearman's rank value(s) you found previously.

What conclusions can you draw from this. How certain are you about the results?

Students can use the work done in this chapter to run a hypothesis test. They will need to decide on and justify their own significance level and this will obviously affect the answer to the question about how certain they are in the results.

Extension

What other experiments or data collection exercises can you think of that will lend themselves to a statistics-based exploration that will result in a hypothesis test like this one. Use the examples and questions in this chapter to give you some inspiration and then design what you will need to do to conduct it.

Students can use this as a review of the chapter but also as a gateway to a possible exploration. By designing an experiment and then being given an opportunity to criticize their own approach (or even that of a classmate) will help them to reflect. They should be reminded that their experiments and data collection will not be perfect – even professional scientists and mathematicians need to be aware of bias or the assumptions and simplifications they have made in order to progress. This does not mean that the exploration is not worth doing

15 Optimizing complex networks: graph theory

Essential understandings

Geometry and trigonometry allow us to quantify the physical world, enhancing our spatial awareness in two and three dimensions. This branch provides us with the tools for analysis, measurement and transformation of quantities, movements and relationships.

Content-specific conceptual understandings

This chapter leads to the following content-specific conceptual understandings listed in the subject guide:

- Graph theory algorithms allow us to represent networks and to model complex real-world problems.
- Matrices are a form of notation which allows us to show the parameters or quantities of several linear equations simultaneously.

We have taken these suggested content-specific conceptual understanding statements and, as recommended in the Teacher Support Manual, developed our own conceptual understandings. Students are led step-by-step through the investigations to arrive at one or more of these conceptual understandings:

Conceptual understandings	Investigation
For an Eulerian circuit to exist all vertices must be an even degree. For an Eulerian trail to exist there must be two odd degree vertices where the two vertices indicate the starting and finishing points.	Investigation 3
The solution to the Chinese postman problem represents the walk of minimum weight that goes along every edge at least once.	Investigation 5

Syllabus sections covered in this chapter:

- AHL3.14
- AHL3.15
- AHL3.16

Cognitive academic language proficiency

The academic language used in this chapter is listed as "microconcepts" at the start of the chapter. Moreover, when specific terminology is introduced it is defined clearly and then used in context to deepen students' understanding.

Cognitive activators

The stimulus questions, developing inquiry skills, before you start activities, investigations and modelling activities are cognitive activators. They get students ready to learn and engage with the subject, providing opportunities for collaborative and peer-to-peer learning, and to develop of inquiry, investigative and modelling skills.

Digital resources

Prior learning support	Animated worked example	GDC skills and support	Additional exercises
Page 691	Page 709: Example 6 Page 710: Example 7 Page 716: Example 9 Page 723: Example 11	NA	Pages 697, 706, 712, 718, 727

Assessment opportunities

End of chapter test	Mixed review exercise	Exam practice
Page 728	Page 729	Page 733

Developing inquiry skills

After initial discussions the questions can be left to the end of the chapter when it would be useful revision of the techniques covered.

15.1 Constructing graphs

TOK

Have you heard of the Seven Bridges of Königsberg problem?

Königsberg is now Kaliningrad in Russia.

Do all mathematical problems have a solution?

This is an opportunity for students to try the Seven Bridges of Königsberg problem. It is a historically notable problem in mathematics. Its negative resolution by Leonhard Euler in 1736 laid the foundations of graph theory and prefigured the idea of topology

Reflect: How do you construct a graph from information given in a table?

Answer: Show the vertices and the connections between them. If weights are given in the table, show them on the graph.

What information can be readily obtained from a graph that cannot easily be seen in a table?

Answer: A graph allows the connections between vertices to be easily seen.

A weighted graph gives extra information about the connections between the vertices in the form of a number, for example, distances between town, cost of flights, etc.

Reflect: When is it appropriate to use directed or undirected graphs?

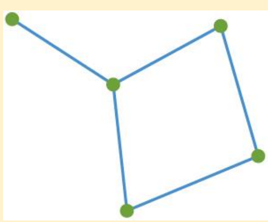
Answer: Directed graphs represent asymmetrical relationships between vertices, indicating direction and apply to many real-life situations such as road networks, hyperlinks connecting webpages and financial trade.

Investigation 1

Though the Handshaking Theorem is not in the syllabus this investigation provides students the opportunity to practise drawing graphs and to discover for themselves one of the famous theories in Graph Theory

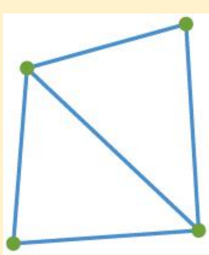
1 **b** There are 6 handshakes.

2 **a**

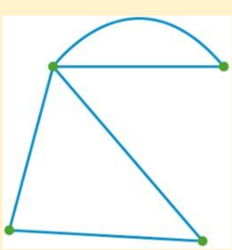


b not possible

c



d



3 The sum of the degrees of the vertices is twice the number of edges.

4 The sum of the degrees comes to an odd number this is not possible as it needs to equal twice the number of edges.

5 Because of the vertices were people and the edges handshakes between people the number of edges would be the number of handshakes and the degree of each vertex would be the number of handshakes made by that person.

6 The sum of in degrees = sum of out degrees = the number of directed edges (if each undirected edge is represented by two directed edges))

15.2 Graph theory for unweighted graphs

TOK

To what extent can shared knowledge be distorted and misleading?

Answer Shared knowledge consists of belief and practices which are communicated to other people and might well need translation by the receiver. The receiver might need enough skill to understand what is being transmitted and there are many famous cases of mistaken knowledge such as the charge of the light brigade.

An individual's knowledge claims might be distorted by peer pressure, fantasy or bias.

Reflect: What can you deduce about the adjacency matrix of a simple graph from the above definition?

Answer: This means the adjacency matrix for a simple graph will have 1 or 0 in each cell and 0s on the leading diagonal.

Investigation 2

Conceptual understanding:

The powers of an adjacency matrix give the number of paths of a given length between two points in a graph and provide an efficient method for determining the minimum length of a path.

1 a i 1 (ABC)

ii 3 (CAC, CBC, CDC)

iii 4 (ACDC, ACBC, ACAC, ABAC)

iv 0

2 a

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

b

$$M^2 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \qquad M^3 = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \\ 4 & 4 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{pmatrix}$$

c

3 Any directed graph can be used to verify this result

4 **Factual:** How do you find the number of walks of length n between two vertices in a graph?

> **Answer:** The number of walks of length n from vertex i to vertex j is the entry in the ith row and the jth column of the nth power of the adjacency matrix.

5 The number of walks of length 1 plus the number of walks of length 2 plus the number of walks of length 3 = $M + M^2 + M^3 = \begin{pmatrix} 4 & 5 & 6 & 2 \\ 5 & 4 & 6 & 2 \\ 6 & 6 & 5 & 4 \\ 2 & 2 & 4 & 1 \end{pmatrix}$

6 **Factual:** What can be calculated from the powers of an adjacency matrix?

> **Answer:** The number of paths of a given length between two points in a graph

7 **Factual:** How can you use the powers of an adjacency matrix to find the minimum length of path between two vertices?

> **Answer:** Sum the powers from 1 to k. The first value of k for which the ij entry is non-zero gives the length of the shortest path between i and j

8 **Conceptual:** How does the adjacency matrix allow you to analyse the paths between two points on a graph?

> **Answer is the TU:** The powers of an adjacency matrix give the number of paths of a given length between two points in a graph and provide an efficient method for determining the minimum length of a path.

Reflect: How might you work out the maximum number of steps necessary to move between any two vertices on a graph?

How can we use powers of the adjacency matrix to determine whether or not a graph is connected?

> **Answer:** The powers of an adjacency matrix give the number of paths of a given length between 2 points in a graph and provides an efficient method for determining the minimum length of a path.

TOK

Matrices are used in computer graphics for three-dimensional modelling.

How can this be used in real-life situations in other areas of knowledge?

> Answer This is an opportunity to explore the other areas of knowledge apart from mathematics such as 3D printing in the arts and to advance diagnoses of health conditions in the natural and human sciences.

Reflect: How do you construct a transition matrix for a given graph?

Why might a random walk indicate the most important sites on a graph?

How can the steady state probabilities be used to rank lists?

> **Answer:** The steady state vector of a transition matrix gives the proportion of “time spent” at each vertex during a random walk. This has applications in ranking lists, including the order of results produced by internet searches.

15.3 Graph theory for weighted graphs: the minimum spanning tree

TOK

Is imagination more important than knowledge?

Have students express their thoughts on what is imagination.

"Imagination is the beginning of creation. You imagine what you desire, you will what you imagine and at last you create what you will." George Bernard Shaw

Albert Einstein thought so. He said: "I'm enough of an artist to draw freely on my imagination, which I think is more important than knowledge.

Knowledge is limited. Imagination encircles the world."

When you see your students share the knowledge they have learned from you, don't you feel proud?

Now, when you see their imagination use that knowledge and take a step further, that's amazing.

Have students write about which they think is the more important, and why.

Reflect: What would be the least-weight way to connect all the vertices in a graph?

Why is Prim's a better algorithm than Kruskal's for a large graph or when the information is given in a table?

How should the algorithm be changed if extra restrictions are added, for example if there must be a direct connection between A and B?

Answer: Minimum spanning trees give the least-weight way to connect all the vertices in a graph.

15.4 Graph theory for weighted graphs: the Chinese postman problem

Investigation 3

Conceptual understanding:

For an Eulerian circuit to exist all vertices must be an even degree.

For an Eulerian trail to exist there must be two odd degree vertices where the two vertices indicate the starting and finishing points.

Additional notes

The purpose of the investigation is for the student to discover the conditions for the existence of an Eulerian circuit and an Eulerian trail.

1 As all the vertices are even it is very easy to find four different routes, starting at any of the vertices

2 **a** Any way of drawing the graph will result in ending at the same vertex as they began.

b All the vertices are of even degree.

c They may need a general explanation of necessary and sufficient conditions if they have not met the idea before.

i **Conceptual:** From the above results conjecture a sufficient condition for being able to traverse all the edges in a graph exactly once, ending at the vertex you began at.

Answer: For an Eulerian circuit to exist all vertices must be an even degree.

ii A few drawings of graphs whose vertices are all of even degree should verify their conjecture.

iii They could draw a few graphs with some vertices of odd degree to see what would happen. A justification might be that whenever you "go into a vertex" you need to come out along a different edge (and you must leave and return to the starting vertex) so all the vertices have to be of even degree.

3 In order to draw the graph it is necessary to start at one of the odd vertices and finish at the other (D and C)

4 **a** There are two odd vertices. The route always goes from one of these to the other.

b **i** **Conceptual:** Conjecture necessary conditions for being able to draw a graph without taking your pen off the paper, and ending at a different vertex to the one at which you began.

Answer: For an Eulerian trail to exist there must be two odd degree vertices where the two vertices indicate the starting and finishing points.

iii Justification: A trail will pass in and out of all vertices except the first and the last, so all these much be of even degree. The first and last must be odd degree as they go out/in without returning.

Investigation 4

The purpose of this investigation is for the students to experiment with trying to find a solution to the CPP. The question is set in a real-life context. There are only two odd vertices. They might be able to make the connection with the previous result concerning Eulerian trails.

1 The students have not yet covered the methods but the actual minimum time that they may or may not reach are:

$279 + 0.25 \times 19 = 283.75$ min

2 There are many answers to this, but each will begin and end at the estate office and repeat the route.

3 Middle school, senior school, cafeteria.

5 In question **5** there are four odd vertices so it is possible the students will not reach a solution but should be able to see that the answer is not simply to take 20 off the total.

$259 + 0.25 \times 41 = 269.25$ min so time saved is 14.5 min.

TOK

What is most important in becoming an intelligent human being: nature or nurture?

This popular debate considers whether the environment or family genes are more important in academic development.

Do intelligent parents always produce intelligent students? What happens if an intelligent child is born in poverty and isolation?

Have students debate on one side or the other and conclude. Try to bring in the phrase "tabula rasa" which means a clean slate, or an absence of preconceived ideas. There are many online videos for teachers to show as a stimulus.

Reflect: How do the conditions for the existence of an Eulerian circuit or trail help in solving the Chinese postman problem?

Answer: The conditions for the existence of an Euler circuit or trail allow us to work out which edges need to be repeated in the Chinese postman problem.

Investigation 5

Conceptual understanding:

The solution to the Chinese postman problem represents the walk of minimum weight that goes along every edge at least once.

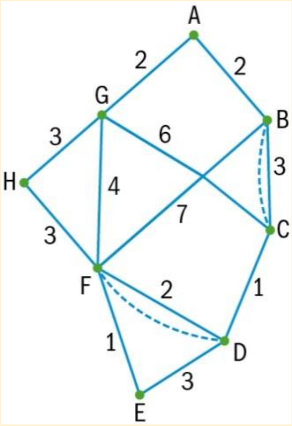
1 The four vertices are B, C, D, F.

2 BC 3 DF 2

BD 4 CF 3

BF 6 CD 1

3



There are many possible routes to take, but whichever one is chosen the edges BC and DF need to be repeated.

4 Total weight is the weight of all the edges plus the repeated routes so 37 + 5 = 42.

5 If you can start and finish at two of the vertices of odd degree then there would only be a need to repeat the edges connecting the other two vertices of odd degree. You therefore need to choose the vertices that will make this as small as possible. In this question the route of least weight is 1 between C and D. Hence choose to begin at B and finish at F, or vice versa.

6 **Factual:** How many extra possible routes do we need to consider when the graph has four vertices of odd degree?

Answer: 3

7 **Conceptual:** How does an understanding of an Eulerian trail help determine how the algorithm should be adapted if the trail does not have to begin and end at the same vertex?

Answer: If the route does not have to begin and end at the same vertex then there can be two odd vertices which do not have to be paired and the route should begin at one and end at the other.

What does the solution to the Chinese postman problem represent?

Answer: The solution to the Chinese postman problem represents the walk of minimum weight that goes along every edge at least once.

15.5 Graph theory for weighted graphs: the travelling salesman problem

Investigation 6

This is a short investigation and could be done as a starter for a lesson.

It does require knowledge of factorial notation and how to calculate factorials on a GDC.

The purpose of the investigation is to draw attention to the fact that the number of Hamiltonian cycles increases dramatically as the number of vertices increases. It therefore very quickly becomes impossible to find the Hamiltonian cycle of least weight by checking the weight of each of the cycles.

1 **a** 3

b 12

c 60

2 **Factual:** Write down a formula for the number of Hamiltonian cycles in a complete graph with n vertices.

Answer: From the calculations in question **1,** students should be able to spot the formula as

$$\frac{n-1\,!}{2}$$

3 This can easily be done by experimenting with different values on a calculator. The answer of just 60 is usually a surprise.

Investigation 7

The purpose of the investigation is for the student to appreciate some of the difficulties in trying to solve the TSP and to begin to develop some strategies for tackling the problem.

1 5

2 It is the sum of the five lowest costs

3 5

4 \$640 and \$900

5 \$773

6 Starting at B, C and D all give a cost of \$722, beginning at E gives \$741.

7 Recommended route ACBDEA or reverse.

8 A cheaper route is AECBDA which costs \$676.

Quite likely to be cheapest as not much above the minimum.

Reflect: What algorithms can you use to find an upper bound for the TSP for a particular graph?

Answer: The weight of any cycle will be an upper bound. A cycle can be found by inspection or by using the nearest neighbor algorithm.

TOK

How long would it take a computer to test all of the Hamiltonian cycles in a complete, weighted graph with just 30 vertices?

Can students use intuition or reasoning to come up with the best answer?

Reflect: How do the algorithms used in the travelling salesman problem help address the fact that it is often not possible to test all the possible routes?

Answer: The algorithms used in the travelling salesman problem give upper and lower bounds for the minimum weight of a cycle which visits every vertex in a graph.

How can you make sure that the nearest neighbour algorithm and the deleted vertex algorithm will give the upper and lower bounds for a practical TSP?

Answer: Forming a table of least distances allows the algorithms for the classical travelling salesman problem to be used in a solution of a practical travelling salesman problem.

TOK

Could we ever reach a point where everything important in a mathematical sense is known?

What does it mean to say the travelling salesman problem is “NP hard”.

Is there any limit to our mathematical knowledge?

NP-hard problem means there is no known algorithm that can solve it.

NP-hard problems are problems for which there is no known polynomial algorithm, so that the time to find a solution grows exponentially with problem size. Although it has not been definitively proven that there is no polynomial algorithm for solving NP-hard problems, many eminent mathematicians have tried and failed.

Choosing the right path

Approaches to Learning/learner profile: Communication, Research, Reflection, Creative Thinking

Exploration Criteria: Presentation (A), Mathematical Communication (B), Personal Engagement (C), Reflection (D);

IB Topic: Graph Theory

Introduction

Graph Theory is an area of mathematics that can be used to solve real-life problems, and this is particularly interesting and engaging when this problem is relevant to a student’s own situation and/or a question devised by the student themselves. This task gives students the opportunity to consider the work they have done in the chapter (or in the extension work related to what they have done in the chapter and devise an exploration outline. To do this they will need to consider:

- writing a clear aim (A)
- how the data will be displayed and what terms need to be defined (B)
- how the problem relates to their own circumstances and the research they will need to do to collect the data (C)
- and how they reflect on the assumptions and simplifications that have been necessary to make in order to answer the aim (D).

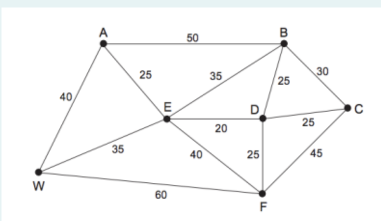
In this chapter you have looked at graph theory problems on the following:

- Minimum spanning trees
- Hamiltonian paths and cycles (the travelling salesman)
- Eulerian trails (the Chinese postman)

(You do not need to answer the following questions – they are examples to provide inspiration!)

Minimum spanning trees

Example question: How many different spanning trees can you find for this graph? What is the minimum length spanning tree?

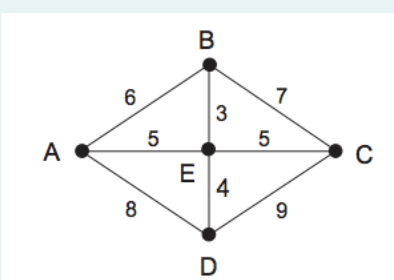


Potential applications:

- Designing a computer network so that it uses the least amount of cabling
- Ensuring that a set of roads in a region stay open in extreme weather for the minimum cost but allowing transport between all villages
- Minimising the cost of phoning people in different countries to pass a message that everyone must get when the cost of calls between the different members of the group are known.

Hamiltonian paths and cycles (the travelling salesman)

Example question: Find the minimum length Hamiltonian cycle on this graph

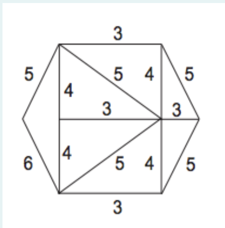


Potential applications:

- Determine the ranks of a team in a competition where each team plays every other team
- Finding the minimum distance for a health inspector to tour around all the restaurants in a town.

Eulerian trails (the Chinese postman)

Example question: Find the minimum length Eulerian circuit on this graph.



What if you did not need to return to the starting vertex?

Potential applications:

- A highways department inspecting a network of roads for fallen trees after a storm in the minimum amount of time/distance.
- Marking out the lines on a football pitch with the shortest distance walked possible.

What real-life situations could you use these to solve?

These real-life applications are particularly interesting and engaging when this problem is relevant to your own situation.

In order to solve a real-life problem it is first necessary to clearly define what it is that you are wishing to find – the shortest path? an Eulerian circuit? And consider the method or algorithm you will follow to make this possible. This is the aim of what you are trying to do.

Then identify what the vertices and edges of the graph represent and find, record or collect the information required to put on this graph. This could be, perhaps, the weightings on edges of the graph, the number of edges that there are between different vertices and the direction of these paths if relevant. You will also need to be able to draw a representation of the graph, perhaps using technology.

You might also reflect on the accuracy and reliability of these values you have found and any assumptions or simplifications it has been necessary to make in order to collect them and finally answer the question. You might also consider what extensions you could solve based on what you have found out so far.

Your task

Choose one of the problems here and think of a situation that is relevant to you that you could use this process to solve.

Based on this try to give a brief answer to each of the following questions.

- What real-life issue are you going to try to address?
- What is the **aim** of this exploration?
- What **personal** reason do you have for wanting to do this exploration?

The above three questions set out what the student would intend to explore and why. The aim is crucial in order to ensure the exploration is concise and complete. If the situation is relevant to the student and local then it might even be possible that they can test out the solution they find.

- What **data** will you need to collect or find?
- What **research** will you need to do for this exploration?
- What **sources of information** will you use?

The above three questions ensure that the student has considered how they will collect the necessary data. It may be important to limit the size of the graph that they are looking at otherwise calculations can become very unwieldy.

- What **definitions** will you need to give?
- What possible **representations** will you need to include?
- How will you **draw** these?
- What **technology** will you require?

Students will need to be able to draw out tables and graphs to represent what they have. They can consider different drawing tools for the graphs. There is free software available that can make drawing graphs easier. Students will also need to decide what they need to define and what notation to use and to ensure this is consistent. Some of the key terms (e.g. Hamiltonian, Eulerian, spanning tree, etc. may need some brief definition).

- What **assumptions** or **simplifications** have you needed to make in this exploration?

This is key. Students should look to reflect critically. For example: If they are considering time to travel around a graph, they may be assuming constant speed and consider whether this is practical; Or distance when they may need to consider slight changes in direction when on a path to avoid unexpected obstacles; If they are "visiting" the vertices do they consider the length of time spent at these.

- What possible **extensions** could there be to your exploration?

As they are working through a problem they may consider extensions – say how they can eliminate some of the simplifications and then build these in to subsequent solutions – or expanding the problem beyond the previous set-up with a change in "rules", etc.

- Are there any **further information/comments** that may be relevant with regards to your exploration?

These types of questions could form the basis of writing an outline for an exploration (and not just in graph theory but in any topic)

Extension

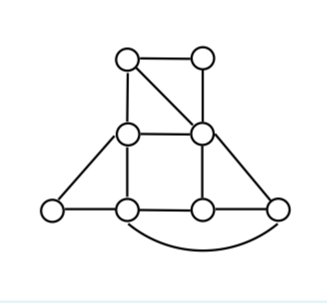
There are also other types of problems in graph theory that have not been covered in this chapter. In order to do these you will need to do some research of the topics. You can then make an outline as above.

These would be considered commensurate with the syllabus and you would be learning new maths which could contribute to criterion C, personal engagement.

Here are two examples on vertex coloring and domination. What real-life situations could you use these to solve?

Vertex coloring

Example question: Find the chromatic number (the smallest number of colors needed to color the vertices of so that no two adjacent vertices share the same color) of the following graph:

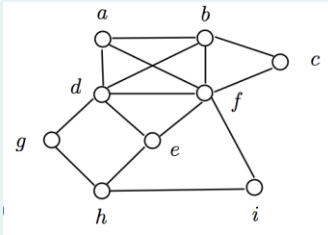


Potential applications:

- Scheduling activities at a school where some students are members of more than one activity.
- Resolving a conflict between radio stations where frequencies would interfere with each other if the stations were too close
- Coloring a map such that no two regions that share a border are painted the same color so that the boundaries can be visualized easily.

Domination

Example question: Find the domination number of the following graph. (The **domination number** of a graph is the size of a smallest **dominating set** which is a set of vertices of the graph such that all vertices have a neighbor in the set.)



Potential applications:

- Placing the smallest number of security guards in a region so that every bank is protected
- Determining the smallest number of stops a school bus must make to drop off all the students close to their homes
- Constructing the least number of mobile phone masts in an area to ensure coverage.